Decision Making and Games with Vector Outcomes*

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June 30, 2019

Abstract

In this paper, we study decision making and games with vector outcomes. We provide a general framework where outcomes lie in a real topological vector space and the decision maker's preferences over outcomes are described by a preference cone, which is defined as a convex cone satisfying a continuity axiom. Further, we define a notion of utility representation and introduce a duality between outcomes and utilities. We provide conditions under which a preference cone is represented by a utility and is the dual of a set of utilities. We formulate a decision-making problem with vector outcomes and study optimal choices. We also consider games with vector outcomes and characterize the set of equilibria. Lastly, we discuss the problem of equilibrium selection based on our characterization.

Keywords: Decision making, Duality, Games, Incomplete preferences, Utility representation, Vector outcomes.
JEL Classification: C02, C72, D01.

1 Introduction

In many real-world decision-making or game situations, the outcome of a choice involves different attributes that can be evaluated separately, but it is difficult to aggregate them into a single utility measure. Zeleny (1975) provides several examples of such situations. If we are to analyze these situations formally, it is natural to work with models where outcomes are vectors and agents have incomplete preferences over outcomes. The goal of this paper

*I am grateful to two anonymous referees and the participants in the Yonsei–Kyoto–Osaka Economic Theory Workshop for helpful comments. I acknowledge the hospitality of the Institute of Economic Research at Kyoto University, where part of this research was conducted. This work was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2016S1A5A2A02925750).

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is to introduce a general framework for studying decision making and games with vector outcomes and to present some fundamental results.

In order to motivate our study from a theoretical point of view, we introduce the setting and the main result of Shapley (1959).[1] He considers a two-player zero-sum game with vector payoffs or outcomes.[2] Each player has a finite number of pure strategies, and the outcome produced when players 1 and 2 choose pure strategies i and j, respectively, is denoted by v_{ij}. Each outcome lies in some Euclidean space $\mathbb{R}^m$ and represents player 1's gains in different commodities. Let $V = [v_{ij}]$ be the matrix of outcomes. When the players use mixed strategies, say p for player 1 and q for player 2, the outcome can be written as $pVq := \sum_i \sum_j p_i v_{ij} q_j$, which also lies in $\mathbb{R}^m$.

Shapley (1959) uses two partial orders defined on $\mathbb{R}^m$, the *strong* partial order $\succ_s$ and the *weak* partial order $\succ_w$.[3] For any two vectors x and y in $\mathbb{R}^m$, we write $x \succ_s y$ if $x_k \geq y_k$ for all $k = 1, \ldots, m$ and $x_k > y_k$ for some $k = 1, \ldots, m$, while we write $x \succ_w y$ if $x_k > y_k$ for all $k = 1, \ldots, m$. Next, he notes the following duality between the two partial orders:

$$x \succ_w 0 \text{ if and only if } x \cdot y > 0 \text{ for all } y \succ_s 0, \tag{1}$$

$$x \succ_s 0 \text{ if and only if } x \cdot y > 0 \text{ for all } y \succ_w 0, \tag{2}$$

where $x \cdot y$ denotes the dot product of the two vectors x and y in $\mathbb{R}^m$. Based on the two partial orders, Shapley (1959) defines two notions of equilibria. A mixed strategy profile (p^*, q^*) is a *strong equilibrium* of the zero-sum game V if there is no p such that $pVq^* \succ_s p^*Vq^*$ and there is no q such that $p^*Vq^* \succ_s p^*Vq$. Similarly, (p^*, q^*) is a *weak equilibrium* of V if there is no p such that $pVq^* \succ_w p^*Vq^*$ and there is no q such that $p^*Vq^* \succ_w p^*Vq$.

Given any vector $\gamma \in \mathbb{R}^m$, we can consider a standard zero-sum game with scalar payoffs where $\gamma \cdot v_{ij}$ is the payoff to player 1 when players 1 and 2 choose pure strategies i and j, respectively. Let γV denote the scalar payoff function. Shapley's (1959) main result characterizes the two types of equilibria of a zero-sum game with vector payoffs by using Nash equilibria of related games with scalar payoffs. He shows that (p^*, q^*) is a strong equilibrium of the two-player zero-sum game V with vector payoffs if and only if it is a Nash equilibrium of the game $(\gamma V, -\delta V)$ with scalar payoffs for some $\gamma \succ_w 0$ and $\delta \succ_w 0$,

[1] Zhao (2018) recognizes Shapley's (1959) work as one of "three little-known and yet still significant contributions of Lloyd Shapley" and writes that "MOG (multiobjective game) is virtually unknown in today's economics literature and its great potentials remain to be explored."

[2] In this paper, we use the terms "payoffs" and "outcomes" interchangeably to mean the result of a choice. We later introduce "utilities" that represent an agent's preferences over vector payoffs/outcomes. In game theory (especially when dealing with scalar payoffs), payoffs are sometimes interpreted as a utility measure. Thus, to avoid potential confusion between payoffs and utilities, we use the term "outcomes" more frequently than "payoffs."

[3] See Aumann (1962, Sec. 3) for an explanation of the terminology. The strong order is also called the Pareto order in the literature.

and that (p^*, q^*) is a weak equilibrium of V if and only if it is a Nash equilibrium of $(\gamma V, -\delta V)$ for some $\gamma \succ_s 0$ and $\delta \succ_s 0$.

Note that a strong equilibrium is defined with the strong order $\succ_s$ while it is characterized with the weak order $\succ_w$. Similarly, a weak equilibrium is defined with $\succ_w$ while it is characterized with $\succ_s$. The characterization stems from the duality between the two orders, as given in (1) and (2). The main objective of this paper is to investigate the characterization of equilibria and the duality between the two orders in a more general context. In fact, the main result of Shapley (1959) has been generalized or modified to more general settings in the existing literature. First, Shapley (1959) mentions that his result can be extended to two-player non-zero-sum games. Aumann (1962) generalizes Shapley's (1959) result by allowing a large class of preferences and mentions that the result can be further extended to n-player non-zero-sum games. Bade (2005) studies games with incomplete preferences. In a general setting, she relates equilibria of a game with incomplete preferences to those of games with complete preferences. She also provides a characterization of equilibria in a specific setting where outcomes lie in Euclidean spaces and players' preferences over outcomes are given by the strong order. Mármol et al. (2017) supplement Bade's (2005) results by considering the weak order as well. They utilize a duality relationship when deriving a characterization of equilibria.

Our framework is general in the following aspects. First, we allow for outcomes in topological vector spaces, which include infinite-dimensional spaces. In contrast, the aforementioned existing works focus on finite-dimensional outcome spaces. Second, we consider a class of preferences described by a convex cone satisfying a continuity axiom, which includes the strong and weak orders as special cases and corresponds to the class assumed in Aumann (1962). Third, we allow infinite action spaces and study pure strategy equilibria, as in Bade (2005). This covers the case of finite pure strategy spaces and mixed strategy equilibria, which is the focus of Shapley (1959) and Aumann (1962). Last, while characterizing optimal choices and equilibria based on duality, we consider more general forms of preferences and sets of utilities than those assumed in Mármol et al. (2017). These features of our framework enable us to analyze a wide range of decision-making and game scenarios with vector outcomes. For example, when comparing two probability distributions, a decision maker may prefer one that first-order stochastically dominates the other; such preferences can be captured in our framework. Moreover, when outcomes involve infinitely many states or time periods, we may need to use an infinite-dimensional vector space as the outcome space.

We take a real topological vector space as the outcome space and its topological dual space as the utility space. An agent's preferences over outcomes are described by a convex cone in the outcome space that satisfies a continuity axiom, and such a cone is called a preference cone. We define the notion of utility representation of a preference cone as well as a duality between a set of outcomes and a set of utilities, generalizing the duality in (1) and (2). The basic properties of dual sets (Lemma 1) provide us with justifications for using a preference cone. Moreover, describing preferences by a convex cone is quite common in the vector (or multicriteria) optimization literature (see, for example, Boţ et al., 2009; Jahn, 2011; Yu, 1974). We investigate conditions under which a given preference cone admits a utility representation (Theorem 1) and can be completely described by a set of utilities (Theorem 2). These two questions are addressed in Aumann (1962) with a finite-dimensional outcome space. There are related results in the vector optimization and economics literature, and we mention them after the respective theorems. In particular, Dubra et al. (2004) examine the problem of representing a preference relation by a set of utilities in a more structured context, and we compare their setup with ours in Remark 1 of Section 2.2.

Next, we characterize optimal choices in a decision-making problem with vector outcomes by considering optimal choices in related decision-making problems where outcomes are scalarized by utilities representing the decision maker's preference cone (Theorem 3). This result is analogous to scalarization results in vector optimization (see, for example, Boţ et al., 2009; Ehrgott, 2005; Jahn, 2011). Then, we extend this result to games and relate equilibria of a game with vector outcomes to Nash equilibria of related games where each player's payoffs are scalarized by utilities representing his preference cone (Theorem 4). Our characterization of equilibria generalizes that of Shapley (1959) as well as some results in the subsequent studies by Aumann (1962), Bade (2005), and Mármol et al. (2017). According to our characterization result, once players select their utilities, playing a Nash equilibrium of the scalarized game yields an equilibrium of the original game with vector payoffs. Selecting a scalarization considerably reduces the multiplicity of equilibria, and thus scalarization can serve as an equilibrium selection tool. We argue that players may coordinate on a scalarization that produces desirable properties of the scalarized game (such as those of zero-sum games or potential games). Given that a game with vector payoffs typically has a huge set of equilibria, various approaches to refine equilibria have been proposed in the literature. For example, Ghose and Prasad (1989) consider equilibria with Pareto optimal security levels using the concept of "Pareto optimal security strategies;" De Marco and Morgan (2007) focus on equilibria that are stable with respect to perturbations on the scalarization; and Hamel and Löhne (2018) introduce a refinement of equilibria based on set relations. Our selection argument will complement these existing refinements.

The literature on games with vector payoffs[4] began with Blackwell (1956) and Shapley

[4]In the literature, games with vector payoffs are also called multicriteria games, multiobjective games, games with multiple goals, games with multiple payoffs, vector-valued games, etc.

(1959). While decision making with vector payoffs has been studied extensively, especially in the vector optimization literature, games with vector payoffs have been relatively less explored.[5] However, in recent years, the literature has seen an increasing number of studies on games with vector payoffs, and we mention some recent works, besides the aforementioned works of Mármol et al. (2017) and Hamel and Löhne (2018). Rettieva (2017) proposes a solution concept for dynamic multicriteria games based on the Nash bargaining solution. Puerto and Perea (2018) study minimax and Pareto-optimal security payoff vectors for general multicriteria zero-sum games. Kokkala et al. (2019) and Sasaki (2018) define rationalizability and examine its properties in the context of games with incomplete preferences and those with vector payoffs. Zapata et al. (2019) consider scalarizations using a weighted minimum function instead of a linear function and study the relationship between equilibria of a game with vector payoffs and those of related scalarized games.

The rest of this paper is organized as follows. We introduce preferences and utilities in Section 2. We study decision-making problems with vector outcomes in Section 3 and games with vector outcomes in Section 4. We conclude in Section 5.

2 Preferences and Utilities

2.1 Preferences

We consider a scenario where a decision maker chooses an alternative and the chosen alternative leads to an outcome. An outcome lies in a topological vector space X over the real field. The decision maker has preferences over outcomes, and they are described by a binary relation $\succ$ on X. For any outcomes x and y, $x \succ y$ means that the decision maker strictly prefers x to y. We impose the following assumptions on $\succ$.

(A1; Translation Invariance) For any $x, y, z \in X$, $x \succ y$ implies $x + z \succ y + z$.
(A2; Nontriviality) There exists $x, y \in X$ such that $x \succ y$.
(A3; Irreflexivity) $x \succ x$ does not hold for any $x \in X$.
(A4; Transitivity) For any $x, y, z \in X$, $x \succ y$ and $y \succ z$ implies $x \succ z$.
(A5; Homotheticity) For any $x, y \in X$ and $\alpha > 0$, $x \succ y$ implies $\alpha x \succ \alpha y$.
(A6; Continuity) If there is a net $\{x^d\}$ with limit $x \in X$ such that $x^d \succ 0$ for all d, then $0 \succ x$ does not hold.

If $x \succ y$, then $x - y$ is an *improvement direction* from y. Under (A1), any improvement direction is independent of starting points, and thus we can describe the decision maker's

[5] See, for example, Boţ et al. (2009), Ehrgott (2005), Jahn (2011), Sawaragi et al. (1985), Tanino and Sawaragi (1979), and Yu (1974) for studies on multicriteria decision making and vector optimization. See also Corley (1985), Nieuwenhuis (1983), Zeleny (1975), Zhao (1991, 2018), and references therein for studies on games with vector payoffs.

preferences by the set of improvement directions, denoted by $W \subseteq X$.[6] That is, if $d \in W$, then $x + d \succ x$ for all $x \in X$. For example, the set of improvement directions generated by the strong order $\succ_s$ on $\mathbb{R}^m$ is $W_s := \{x \in \mathbb{R}^m : x_k \geq 0 \ \forall k = 1, \dots, m\} \setminus \{0\}$, while the weak order $\succ_w$ on $\mathbb{R}^m$ yields $W_w := \{x \in \mathbb{R}^m : x_k > 0 \ \forall k = 1, \dots, m\}$.

With (A1) imposed, the properties of $\succ$ can be translated to those of W as follows. $\succ$ satisfies (A2) if and only if $W \neq \varnothing$. $\succ$ satisfies (A3) if and only if $0 \notin W$. $\succ$ satisfies (A4) if and only if $W + W \subseteq W$. $\succ$ satisfies (A5) if and only if W is a cone.[7] $\succ$ satisfies (A6) if and only if $\overline{W} \cap (-W) = \varnothing$.[8] $\succ$ satisfies (A4) and (A5) if and only if W is a convex cone, while imposing (A3) in addition makes W a blunt convex cone.[9] Note that (A6) implies (A3). Thus, a strict preference relation $\succ$ on X satisfying (A1)–(A6) can be expressed as a set of improvement directions $W \subseteq X$ that is a nonempty convex cone satisfying $\overline{W} \cap (-W) = \varnothing$, and vice versa. We say that $W \subseteq X$ is a *(continuous strict) preference cone* if it is a nonempty convex cone and satisfies $\overline{W} \cap (-W) = \varnothing$. It is easy to see that the two sets W_s and W_w defined above are preference cones.

In the following, we will maintain (A1)–(A6) for the decision maker's preferences and describe them by a preference cone. A binary relation $\succ$ on X is called a (strict) partial order if it is irreflexive and transitive. So if the vector space X is equipped with a preference cone W, we say that X is partially ordered by W.

We use strict preferences as a primitive because indifference does not play a role in our analysis. Alternatively, we can start from weak preferences $\succsim$ and derive strict preferences $\succ$ by defining $x \succ y$ if $x \succsim y$ and not $y \succsim x$. In the vector optimization literature, it is common to begin with a (non-strict) partial order (i.e., a reflexive, transitive, and antisymmetric[10] binary relation) $\succsim$ and associate it with a (weak) preference cone C such that $x \succsim y$ if and only if $x - y \in C$ (see, for example, Jahn, 2011). Since antisymmetry excludes indifference between two distinct outcomes, our approach can cover this alternative approach with $W = C \setminus \{0\}$. Meanwhile, our approach is more general in that it can be applied to some weak preferences that are not antisymmetric. For example, consider $X = \mathbb{R}^2$ and $C = \{(x_1, x_2) \in X : x_1 + x_2 \geq 0\}$. Then the weak preference relation $\succsim$ associated with C is not antisymmetric, but the strict preference relation $\succ$ derived from $\succsim$ satisfies (A1)–(A6) with $W = \{(x_1, x_2) \in X : x_1 + x_2 > 0\}$. Since $x \not\succ y$ and $y \not\succ x$ does not necessarily mean that the decision maker is indifferent between x and y under our assumptions on $\succ$, we deal with the case of incomplete preferences, which is natural in the case of vector

[6] Yu (1974) provides an analysis on the case where the set of improvement directions depends on the starting point.

[7] A set B in a real vector space is a *cone* if for any $x \in B$ and $\alpha > 0$, $\alpha x \in B$.

[8] For any set B in a topological space, $\overline{B}$ denotes the closure of B.

[9] A set B in a real vector space is a *convex cone* if for any $x, y \in B$ and $\alpha, \beta > 0$, $\alpha x + \beta y \in B$. A convex cone B is *blunt* if $0 \notin B$.

[10] A binary relation $\succsim$ is antisymmetric if $x \succsim y$ and $y \succsim x$ implies $x = y$.

outcomes.

2.2 Utilities

We present our concept of utility representation.

Definition 1. A *utility* u *representing a preference cone* $W \subseteq X$ is a continuous linear functional on X such that $x \in W$ implies $u(x) > 0$.

Our concept of utility representation is analogous to that of Aumann (1962) in that we require only one-way implications. This kind of utilities is called Richter–Peleg utility functions in Ok (2002), and it appears frequently in the literature on incomplete preferences.

Let X^* be the topological dual space of X, that is, the set of all continuous linear functionals on X. Note that X^* can be considered as the space of candidates for utilities. So we call X the outcome space and X^* the utility space. For any $Y \subseteq X$, define

$$Y^+ = \{u \in X^* : u(x) > 0 \text{ for all } x \in Y\}. \tag{3}$$

That is, Y^+ is the set of utilities that yield positive values for all outcomes in Y. Also, for any $Z \subseteq X^*$, define

$$^+Z = \{x \in X : u(x) > 0 \text{ for all } u \in Z\}. \tag{4}$$

That is, ^+Z is the set of outcomes that are assigned positive values for all utilities in Z.

The first operator Y^+ in (3) takes a subset in the outcome space and generates a subset in the utility space, while the second operator ^+Z in (4) operates in the opposite direction. We say that $Y \subseteq X$ is the dual of $Z \subseteq X^*$ if $Y = {}^+Z$. Similarly, we say that $Z \subseteq X^*$ is the dual of $Y \subseteq X$ if $Z = Y^+$. When both $Y = {}^+Z$ and $Z = Y^+$ hold, Y and Z are dual of each other. This duality generalizes the duality between the strong and weak partial orders observed in (1) and (2). When $X = \mathbb{R}^m$, X^* is also given by $\mathbb{R}^m$, and the relationships in (1) and (2) can be expressed as $W_w = (W_s)^+ = {}^+(W_s)$ and $W_s = (W_w)^+ = {}^+(W_w)$. Thus, the two sets W_s and W_w are dual of each other, regardless of whether we take W_s or W_w as a subset of the outcome space.

For any set B in a real vector space, let $cone(B)$ be the smallest convex cone containing B, that is,

$$cone(B) = \left\{ \sum_{j=1}^{k} \alpha_j x^j : x^1, \ldots, x^k \in B, \alpha_1, \ldots, \alpha_k > 0, k \in \{1, 2, \ldots\} \right\}.$$

As a preliminary, we present basic properties of the dual sets Y^+ and ^+Z in the following lemma.

Lemma 1. *Let X be a real topological vector space, and let X^* be its topological dual space.*
(i) For any $Y_1 \subseteq Y_2 \subseteq X$ and $Z_1 \subseteq Z_2 \subseteq X^*$, $Y_2^+ \subseteq Y_1^+$ and $^+Z_2 \subseteq {}^+Z_1$.
(ii) For any $Y \subseteq X$ and $Z \subseteq X^*$, $Y \subseteq {}^+(Y^+)$ and $Z \subseteq ({}^+Z)^+$.
(iii) For any $\varnothing \neq Y \subseteq X$ and $\varnothing \neq Z \subseteq X^*$, Y^+ and ^+Z are blunt convex cones, Y^+ satisfies $\overline{Y^+} \cap (-Y^+) = \varnothing$ if X is a normed space, and ^+Z satisfies $\overline{{}^+Z} \cap (-{}^+Z) = \varnothing$.
(iv) For any $Y \subseteq X$ and $Z \subseteq X^*$, $Y^+ = (cone(Y))^+$ and $^+Z = {}^+(cone(Z))$.

Proof. (i) The result is immediate from the definitions.

(ii) Choose any $x \in Y$. Then $u(x) > 0$ for all $u \in Y^+$. This implies $x \in {}^+(Y^+)$, proving $Y \subseteq {}^+(Y^+)$.

(iii) Choose any $u, v \in Y^+$ and $\alpha, \beta > 0$. For any $x \in Y$, $u(x), v(x) > 0$, and thus $(\alpha u + \beta v)(x) = \alpha u(x) + \beta v(x) > 0$. Hence, $\alpha u + \beta v \in Y^+$. Since Y is nonempty and $0(x) = 0$ for any $x \in Y$, the zero function cannot be in Y^+. This shows that Y^+ is a blunt convex cone. Suppose that X is a normed space. Then X^* is also a normed space. If $\overline{Y^+} = \varnothing$, we have $\overline{Y^+} \cap (-Y^+) = \varnothing$, and so we assume that $\overline{Y^+} \neq \varnothing$. Choose any $u \in \overline{Y^+}$. Then there exists a sequence $\{u^k\}$ in Y^+ that converges in norm to u. Choose any $x \in Y$. Since $u^k \in Y^+$, we have $u^k(x) > 0$ for all k. Since $\{u^k\}$ also converges in the weak-* topology to u, we have $u(x) \geq 0$. Suppose that $u \in -Y^+$. Then $-u \in Y^+$, and $(-u)(x) > 0$. But then $u(x) < 0$, which is a contradiction. Hence, we obtain $\overline{Y^+} \cap (-Y^+) = \varnothing$.

(iv) Note that $Y \subseteq cone(Y)$, and thus $(cone(Y))^+ \subseteq Y^+$ by (i). Choose any $u \in Y^+$ and any $x \in cone(Y)$. Then there is $x^1, \ldots, x^k \in Y$ and $\alpha_1, \ldots, \alpha_k > 0$ such that $x = \alpha_1 x^1 + \cdots + \alpha_k x^k$. Then $u(x) = \alpha_1 u(x^1) + \cdots + \alpha_k u(x^k) > 0$ since $u(y) > 0$ for all $y \in Y$. Hence $u \in (cone(Y))^+$, proving $Y^+ \subseteq (cone(Y))^+$.

The results about $Z \subseteq X^*$ can be proven analogously. □

Note that the sets Y and Z do not need to be convex cones in the definitions (3) and (4) as well as in Lemma 1. However, the results in Lemma 1(iii) and (iv) suggest that it is natural to consider convex cones when we study dual sets. Lemma 1(iii) also shows that the continuity assumption (A6) is met by dual sets. For a given preference cone W, W^+ is the set of utilities representing W. In the following theorem, we study conditions for the existence of a utility.

Theorem 1. *Let X be a real topological vector space partially ordered by a preference cone W.*
(i) Suppose that (a) X is locally convex and $\overline{W}$ is locally compact or (b) W is open. Then there is a utility representing W. (That is, $W^+ \neq \varnothing$.)
(ii) If a set $Y \subseteq X$ has $Y^+ \neq \varnothing$, then $\overline{Y} \cap (-Y) = \varnothing$.

Proof. (i) First, suppose that X is locally convex and $\overline{W}$ is locally compact. If $\overline{W} = X$, then $\overline{W} \cap (-W) = -W \neq \varnothing$, a contradiction to our assumption that $\overline{W} \cap (-W) = \varnothing$. Hence, $\overline{W} \neq X$, and there is $\tilde{x} \notin \overline{W}$. By Theorem 2.5 of Klee (1955) applied to $\overline{W}$ and $\{\alpha\tilde{x} : \alpha \geq 0\}$, there exists $u \in X^*$ such that $u(x) > 0 \geq u(\tilde{x})$ for all $x \in \overline{W} \setminus (-\overline{W})$. Since $\overline{W} \cap (-W) = \varnothing$, we have $(-\overline{W}) \cap W = \varnothing$. Hence, $W \subseteq \overline{W} \setminus (-\overline{W})$, and $u \in W^+$.

Next, suppose that W is open. Since $0 \notin W$ and W is nonempty and convex, by the Hahn–Banach separation theorem, there exists $u \in X^*$ such that $u(x) > u(0) = 0$ for all $x \in W$. Thus, $u \in W^+$.

(ii) Suppose that $Y^+ \neq \varnothing$. Choose any $u \in Y^+$. Suppose to the contrary that there is $x \in \overline{Y} \cap (-Y)$. Since $x \in \overline{Y}$, we have $u(x) \geq 0$. Since $x \in -Y$, we have $-x \in Y$ and $u(-x) > 0$. Then $u(x) < 0$, a contradiction. Thus, $\overline{Y} \cap (-Y) = \varnothing$. □

When X is a finite-dimensional real normed vector space, condition (a) in Theorem 1(i) is satisfied. Hence, as shown in Theorem A of Aumann (1962), there is a utility if X is finite-dimensional, and Theorem 1(i) can be considered as a generalization of this result. The hypothesis in Theorem 1(i) is not necessary for the existence of a utility. In fact, Theorem C of Kannai (1963) shows that there is a utility if X is a separable normed vector space, which includes ℓ^p spaces for $1 \leq p < \infty$. For example, consider $X = \ell^1$ and $W = \{x \in X : x \geq 0\} \setminus \{0\}$.[11] Then neither $\overline{W}$ is locally compact nor W is open. In this case, X^* is isomorphic to ℓ^∞, and it can be seen that $u = (1, 1, \ldots)$ is a utility representing W.[12]

There are topological vector spaces that have a trivial topological dual. As an example, consider the space $X = L^p$ with $0 < p < 1$. The only nonempty convex open set in L^p is the entire space (Rudin, 1991, Sec. 1.47). This implies that L^p is not locally convex and that there cannot be an open preference cone. So neither condition (a) nor (b) in Theorem 1(i) can be satisfied. We have 0 as the only continuous linear functional on L^p (i.e., $X^* = \{0\}$), and thus $Y^+ = \varnothing$ for any nonempty set $Y \subseteq X$. The assumption of local convexity in Theorem 1(i) is imposed to exclude spaces like L^p with $0 < p < 1$. Let W be a preference cone, and let $C = W \cup \{0\}$. In the optimization literature, the set $\{u \in X^* : u(x) \geq 0 \text{ for all } x \in W\}$ is called the topological dual cone for C, and the set W^+ is called the quasi-interior of the topological dual cone for C (see, for example, Jahn, 2011, Def. 1.23). Corollary 3.19 and Theorem 3.38 of Jahn (2011) provide alternative sufficient

[11] We use $x = (x_1, x_2, \ldots) \geq 0$ to mean $x_k \geq 0$ for all $k = 1, 2, \ldots$.

[12] Aumann (1962) provides an example of an infinite-dimensional partially ordered vector space without a utility. In his example, X is the set of all infinite sequences of real numbers, and $W = \{x \in X : x \geq 0\} \setminus \{0\}$. Suppose that there is a utility u representing W. Let $\tilde{e}^1 = (1, 0, 0, \ldots)$, $\tilde{e}^2 = (0, 1, 0, \ldots)$, and so on, and let $u_k = u(\tilde{e}^k)$ for all $k = 1, 2, \ldots$. Note that each $u_k > 0$ since $\tilde{e}^k \in W$. Consider $\hat{x} = (1/u_1, 1/u_2, \ldots)$. Then $u(\hat{x})$ is infinite. From this, Aumann (1962) concludes that there cannot be a utility. However, his argument would not work in our context. If we require X to be a topological vector space and a utility to be in X^*, then the conclusion would be that $\hat{x}$ does not belong to X because $v(x)$ is finite for any $x \in X$ and $v \in X^*$.

conditions for the quasi-interior W^+ to be nonempty. Both of them are concerned with locally convex spaces; the latter is close to Theorem C of Kannai (1963) in that it assumes a real separable normed space.

Theorem 1(ii) provides a necessary condition for $Y^+ \neq \varnothing$. This result is already provided in Theorem A of Kannai (1963) but included for completeness of discussion. It can be used to justify our assumption that $\overline{W} \cap (-W) = \varnothing$; without this assumption, there cannot be a utility representing W. Moreover, since $Y^+ = (cone(Y))^+$ by Lemma 1(iv), our assumption that W is a convex cone involves no loss of generality when we investigate whether W^+ is nonempty or not.

Next, we turn to the question of whether a preference cone W can be expressed as ${}^+Z$ for some $Z \subseteq X^*$, that is, whether W is the dual of some set in the utility space. If this is the case, we can describe the preferences W fully by a *set* of utilities. If, in addition, $Z = W^+$ holds (i.e., if W and Z are dual of each other), then it suffices to specify only one of W and Z, as one of them can be completely determined by the other. Thus, in the following theorem, we investigate conditions for the relationship $W = {}^+(W^+)$.

Theorem 2. *Let X be a real topological vector space partially ordered by a preference cone W.*
(i) Suppose that (a) X is locally convex and $\overline{W}$ is locally compact and (b) W is open or $W = \overline{W} \setminus \{0\}$. Then $W = {}^+(W^+)$.
(ii) If Y is a proper subset of X and $Y = {}^+(Y^+)$, then Y is a convex cone and $\overline{Y} \cap (-Y) = \varnothing$.

Proof. (i) Since $W \subseteq {}^+(W^+)$ by Lemma 1(ii), it remains to show the other inclusion. For this purpose, we will choose an arbitrary $\tilde{x} \notin W$ and show that there exists $u \in X^*$ such that $u(x) > 0 \geq u(\tilde{x})$ for all $x \in W$. Then $u \in W^+$ and $\tilde{x} \notin {}^+(W^+)$, establishing ${}^+(W^+) \subseteq W$.

Suppose that (a) holds. First, suppose that W is open. Choose any $\tilde{x} \notin W$. Suppose that $\tilde{x} \in \overline{W}$. Then by the Hahn–Banach separation theorem applied to W and $\{\tilde{x}\}$, there exists $u \in X^*$ such that $u(x) > u(\tilde{x})$ for all $x \in W$. Since W is a cone, we have $u(x) \geq 0$ for all $x \in W$ and thus $u(\tilde{x}) \geq 0$. Since $0 \in \overline{W}$, we have $u(0) \geq u(\tilde{x})$. Hence, it follows that $u(x) > 0 = u(\tilde{x})$, and we are done. Suppose that $\tilde{x} \notin \overline{W}$. Then $\tilde{x} \neq 0$. Since $\overline{W}$ is locally compact in locally convex X, applying Theorem 2.5 of Klee (1955), we obtain $u \in X^*$ such that $u(x) > 0 \geq u(\tilde{x})$ for all $x \in W$, as in the proof of Theorem 1(i).

Next, suppose that $W = \overline{W} \setminus \{0\}$. Choose any $\tilde{x} \notin W$. Then either $\tilde{x} \notin \overline{W}$ or $\tilde{x} = 0$. Consider the case where $\tilde{x} \notin \overline{W}$. Then, as above, there exists $u \in X^*$ such that $u(x) > 0 \geq u(\tilde{x})$ for all $x \in W$. By Theorem 1(i), W^+ is nonempty, and thus by Lemma 1(iii), we have $0 \notin {}^+(W^+)$. This takes care of the case where $\tilde{x} = 0$.

(ii) Suppose that Y is a proper subset of X and that $Y = {}^+(Y^+)$. Suppose that Y^+ is empty. Then $Y = {}^+(Y^+) = X$, a contradiction. Hence, Y^+ is nonempty. Since $Y = {}^+(Y^+)$, Lemma 1(iii) implies that Y is a convex cone and satisfies $\overline{Y} \cap (-Y) = \varnothing$. $\square$

When $W = {}^+(W^+)$, we can describe the preference cone W by the set of utilities representing W. Theorem 2 provides a sufficient condition for $W = {}^+(W^+)$ as well as a necessary one. As $Y = {}^+(Y^+)$ implies that $Y^+ \neq \varnothing$ for any proper subset Y of X, the conditions in Theorem 2 are stronger than the corresponding ones in Theorem 1.

Neither condition (a) nor (b) in Theorem 2(i) is necessary for $W = {}^+(W^+)$. Consider the previous example of $W = \{x \in \ell^1 : x \geq 0\} \setminus \{0\}$, which does not satisfy condition (a). In this example, we have $W^+ = \{u \in \ell^\infty : u \gg 0\}$[13] and $W = {}^+(W^+)$. As another example, consider $W = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 \geq 0\}$. Here, condition (b) is not satisfied, while we have $W = {}^+(W^+)$.[14] However, we cannot guarantee $W = {}^+(W^+)$ without condition (b). For example, consider $W = \{x \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 \geq 0\} \cup \{x \in \mathbb{R}^3 : x_1 > 0, x_2 = x_3 = 0\} \cup \{x \in \mathbb{R}^3 : x_2 > 0, x_1 = x_3 = 0\}$. Then W is a preference cone satisfying condition (a) but not (b). We have $W^+ = \{u \in \mathbb{R}^3 : u_1 > 0, u_2 > 0, u_3 \geq 0\}$ and ${}^+(W^+) = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\} \setminus \{x \in \mathbb{R}^3 : x_1 = x_2 = 0, x_3 \geq 0\}$, which is strictly larger than W. Lastly, consider the two preference cones W_s and W_w, associated with the strong and weak partial orders, respectively, on $\mathbb{R}^m$. As $W_s = \overline{W}_s \setminus \{0\}$ and W_w is open, they satisfy conditions (a) and (b), and we have seen that the relationship $W = {}^+(W^+)$ holds for them.

Theorem 2(ii) provides a necessary condition for $Y = {}^+(Y^+)$. Again, this result justifies our assumption that W is a convex cone satisfying $\overline{W} \cap (-W) = \varnothing$, since without it the relationship $W = {}^+(W^+)$ cannot hold.

Remark 1. Dubra et al. (2004) take an arbitrary compact metric space S as the set of prizes. They consider weak preferences $\succsim$ defined on the set of all Borel probability measures over S, denoted by $\mathscr{P}(S)$, and assume that the potentially incomplete preference relation $\succsim$ satisfies the standard independence and continuity axioms. They take the vector space generated by $\mathscr{P}(S)$ as the outcome space and extend preferences to the outcome space by working with the set of weak improvement directions, $\mathscr{C}(\succsim) := \{\alpha(p-q) : \alpha > 0 \text{ and } p \succsim q\}$. They have the set of all continuous real maps on S (topologized by the sup-norm) as the utility space, which is separable. They point out that the outcome space is equal to the topological dual of the utility space, and they show that $\mathscr{C}(\succsim)$ is a weak*-closed convex cone in the outcome space. That is, they regard the outcome space as the dual of the utility space, whereas we treat the utility space as the dual of the outcome space. Using the Hahn–Banach separation

[13] We use $x = (x_1, x_2, \ldots) \gg 0$ to mean $x_k > 0$ for all $k = 1, 2, \ldots$.

[14] See Aumann (1962, Sec. 7). Aumann (1962) considers $X = X^* = \mathbb{R}^m$, and in his Theorem D, he states that $W = {}^+(W^+)$ if and only if W is the intersection of its open supports. In our setting, the intersection of the open supports of W is just the definition of ${}^+(W^+)$. Hence, in our Theorem 2(i), we present more primitive conditions on W to obtain $W = {}^+(W^+)$.

theorem, they prove an expected multi-utility representation theorem in the sense that

$$p \succsim q \quad \text{if and only if} \quad \int_S v dp \geq \int_S v dq \quad \text{for all } v \in \mathscr{V},$$

for some subset $\mathscr{V}$ of the utility space. Note that the outcome and utility spaces in Dubra et al. (2004) are more structured than those in this paper, and thus they obtain the multi-utility representation theorem without any further assumptions. While both papers use convex cones to describe preferences over outcomes, their continuity axiom is stronger than ours, provided that we adopt the corresponding topology on the outcome space.[15] Moreover, the notions of utility representation by a set differ slightly in the two papers. In this paper, $W = {}^+(W^+)$ means that $x \succ y$ if and only if $u(x) > u(y)$ for all $u \in W^+$, while multi-utility representation in Dubra et al. (2004) implies that $p \succ q$ if and only if $\int_S v dp \geq \int_S v dq$ for all $v \in \mathscr{V}$, with strict inequality for some $v \in \mathscr{V}$. Thus, our notion of utility representation is slightly stronger for strict preferences; however, we impose no requirement that corresponds to $p \sim q$ if and only if $\int_S v dp = \int_S v dq$ for all $v \in \mathscr{V}$, as we do not care about indifference.

When X is a reflexive Banach space,[16] the elements of the second dual X^{**} can be identified by those of X, and thus we can switch the roles of X and X^*. That is, we can think of X^* as the outcome space and X as the utility space. For example, given a set of possible states, we can consider a scenario where an outcome is a prize vector that specifies a monetary reward for each state and a utility is a probability distribution over states, so that the utility of an outcome is computed as the expected value of prizes with respect to the probability distribution. If X is reflexive, we can analyze the opposite scenario where an outcome is a probability distribution over states and a utility is a prize vector. Taking X^* as the outcome space, we can use our results as follows. First, we can write down conditions on X^* and $Z \subseteq X^*$ that are analogous to those in Theorems 1 and 2 to obtain ${}^+Z \neq \varnothing$ and $Z = ({}^+Z)^+$. Second, suppose that $W = {}^+(W^+)$ holds in reflexive X. Since X is a normed space, W^+ satisfies $\overline{W^+} \cap (-W^+) = \varnothing$ by Lemma 1(iii). Moreover, since $W \neq X$, W^+ is nonempty. Then we can take W^+ as a preference cone in X^*, and W can be regarded as the set of utilities in X^{**} representing W^+. In other words, we can also switch the roles of W and W^+ in this case.

[15] The continuity axiom of Dubra et al. (2004) implies that the weak preference cone C is closed. Let $W = C \setminus (-C)$. To show $\overline{W} \cap (-W) = \varnothing$, suppose to the contrary that there exists $x \in \overline{W} \cap (-W)$. Then $x \in \overline{W}$ and $x \in -W$. Since $W \subseteq C$ and C is closed, we have $x \in C$ and $x \in -C$. This implies $-x \in -C$ and $-x \in C$. Then $-x \notin W$, which contradicts $x \in -W$.

[16] See Sec. 4.5 of Rudin (1991) for the definition of reflexive spaces.

3 Decision Making with Vector Outcomes

In this section, we study the decision maker's optimal choices[17] over alternatives. A *decision-making problem with vector outcomes* is defined by a tuple (A, X, f, W), where A is a nonempty set, X is a real topological vector space, f is a function from A to X, and W is a preference cone in X. As before, X denotes the outcome space, while W expresses the decision maker's preferences over outcomes that satisfy (A1)–(A6). The set A denotes the set of alternatives, and the function f describes the relationship between alternatives and outcomes. That is, when the decision maker chooses alternative $a \in A$, the outcome is given by $f(a) \in X$. For any $B \subseteq A$, we define $f(B) = \{f(a) : a \in B\}$. For notational simplicity, we will use $D := (A, X, f)$ so that a decision-making problem can be written as (D, W). The decision-making problem where the decision maker has a utility u on X is written as (D, u).

Let $O(D, W)$ be the set of optimal choices in the problem (D, W), i.e.,

$$O(D, W) = \{a^* \in A : \nexists a \in A \text{ s.t. } f(a) - f(a^*) \in W\}.$$

Similarly, let $O(D, u)$ be the set of optimal choices in the problem (D, u), i.e.,

$$O(D, u) = \{a^* \in A : \nexists a \in A \text{ s.t. } u(f(a)) > u(f(a^*))\}.$$

$O(D, W)$ is the set of alternatives that lead to the maximal elements of $f(A)$ with respect to the partial order induced by W. In other words, $f(O(D, W))$ is the set of maximal elements of $f(A)$. Note that a utility can be interpreted to induce complete preferences on X,[18] while $O(D, u)$ can be regarded as the set of maximizers of the real-valued function $u \circ f$ on A. In the following theorem, we investigate the relationship between $O(D, W)$ and $O(D, u)$. We use the notation $f(A) - (W \cup \{0\})$ to denote the set $\{x - y \in X : x \in f(A), y \in W \cup \{0\}\}$. That is, $f(A) - (W \cup \{0\})$ is the extension of $f(A)$ by adding all outcomes that are less preferred to some outcome in $f(A)$.

Theorem 3. *Let (D, W) be a decision-making problem with vector outcomes. Then*

$$\bigcup \{O(D, u) : u \in W^+\} \subseteq O(D, W). \tag{5}$$

[17] In the multicriteria decision making literature, different terminologies such as Pareto optimal and efficient solutions are used to refer to optimal choices (see Ehrgott, 2005, Table 2.4).

[18] Given a utility u on X, we can define a weak preference relation $\succsim$ on X by $x \succsim y$ if and only if $u(x) \geq u(y)$. Then $\succsim$ is complete (i.e., for any $x, y \in X$, either $x \succsim y$ or $y \succsim x$).

Suppose that $f(A) - (W \cup \{0\})$ *is convex*[19] *and that* $\mathrm{int}(f(A) - (W \cup \{0\}))$ *is nonempty.*[20] *Then*

$$O(D, W) \subseteq \bigcup\{O(D, u) : u \in (\mathrm{int}W)^+\}. \tag{6}$$

Proof. Let $a^* \in O(D, u)$ for some $u \in W^+$. Suppose to the contrary that $a^* \notin O(D, W)$. Then there exists $a \in A$ such that $f(a) - f(a^*) \in W$. Since $u \in W^+$, we have $u(f(a) - f(a^*)) > 0$ and thus $u(f(a)) > u(f(a^*))$. This contradicts $a^* \in O(D, u)$.

Suppose that $f(A) - (W \cup \{0\})$ is convex and that $\mathrm{int}(f(A) - (W \cup \{0\}))$ is nonempty. Choose any $a^* \in O(D, W)$. Then $f(a^*)$ is maximal in $f(A)$ with respect to W. By Lemma 4.7(b) of Jahn (2011), $f(a^*)$ is also maximal in $f(A) - (W \cup \{0\})$. Since W is a cone, $f(a^*)$ cannot be in the interior of $f(A) - (W \cup \{0\})$. Since $f(A) - (W \cup \{0\})$ is convex, so is $\mathrm{int}(f(A) - (W \cup \{0\}))$. Then by the Hahn–Banach separation theorem, there exists $u \in X^*$ such that $u(f(a^*)) > u(x)$ for all $x \in \mathrm{int}(f(A) - (W \cup \{0\}))$. We show that $u \in (\mathrm{int}W)^+$. If $\mathrm{int}W = \varnothing$, then $(\mathrm{int}W)^+ = X^*$, and we obtain $u \in (\mathrm{int}W)^+$. So suppose that $\mathrm{int}W \neq \varnothing$, and choose any $x \in \mathrm{int}W$. Since $f(a^*) \in f(A)$, we have $f(a^*) - x \in f(A) - (W \cup \{0\})$. Since x is an interior point of W, $f(a^*) - x$ is an interior point of $f(A) - (W \cup \{0\})$. Hence, $u(f(a^*)) > u(f(a^*) - x) = u(f(a^*)) - u(x)$, and so $u(x) > 0$. This implies $u \in (\mathrm{int}W)^+$. We have $u(f(a^*)) \geq u(x)$ for all $x \in f(A) - (W \cup \{0\})$. Since $f(A) \subseteq f(A) - (W \cup \{0\})$, we obtain $a^* \in O(D, u)$. $\square$

Theorem 3 provides a sufficient condition for optimal choices, as well as a necessary one under additional assumptions. From Theorem 3, we can obtain a sufficient condition for the existence of optimal choices. If A is a compact subset of a topological space, f is continuous, and W^+ is nonempty, then $O(D, u)$ is nonempty for any $u \in W^+$, and thus $O(D, W)$ is nonempty. Combining the two inclusions in (5) and (6), we obtain

$$\bigcup\{O(D, u) : u \in W^+\} \subseteq O(D, W) \subseteq \bigcup\{O(D, u) : u \in (\mathrm{int}W)^+\}.$$

Collecting all optimal choices of a decision maker with a utility representing W provides a lower bound on the set of optimal choices in the decision-making problem (D, W), while collecting those of a decision maker with a utility representing $\mathrm{int}W$ offers an upper bound. By Lemma 1(i), we have $W^+ \subseteq (\mathrm{int}W)^+$, and thus it is clear that the upper bound is at least as large as the lower bound. Typically, these two bounds will not differ much. In

[19] A sufficient condition for $f(A) - (W \cup \{0\})$ to be convex is that A is a convex subset of a real vector space and f is W-concave in the sense that $f(\alpha a + (1 - \alpha)a') - [\alpha f(a) + (1 - \alpha) f(a')] \in W \cup \{0\}$ for any $a, a' \in A$ and $\alpha \in [0, 1]$.

[20] For any set B in a topological space, $\mathrm{int}(B)$ or $\mathrm{int}B$ denotes the topological interior of B. Obviously, if $f(A)$ or W has a nonempty interior, then $\mathrm{int}(f(A) - (W \cup \{0\}))$ is nonempty. However, it can be nonempty even when both $\mathrm{int}f(A)$ and $\mathrm{int}W$ are empty.

particular, when W is open, we have $W = \mathrm{int}W$, and the two bounds will coincide. This observation leads to the following corollary.

Corollary 1. *Let (D, W) be a decision-making problem with vector outcomes. Suppose that W is open and that $f(A) - (W \cup \{0\})$ is convex. Then*

$$O(D, W) = \bigcup \{O(D, u) : u \in W^+\}. \tag{7}$$

When W is not open, we cannot guarantee the relationship in (7), as pointed out in Shapley (1959). Aumann (1962, Theorem B) provides an alternative sufficient condition by showing that the relationship in (7) holds if $f(A)$ is a convex polyhedron (i.e., the convex hull of finitely many points) in a Euclidean space (see also Arrow et al., 1953, Theorem 1, where they focus on the Pareto order).

Theorem 3 and Corollary 1 are closely related to scalarization results in vector optimization. Any element of $f(O(D, u))$ where $u \in W^+$ is called a properly maximal element of $f(A)$ in the sense of linear scalarization in Boţ et al. (2009, Def. 2.4.12) and an almost properly maximal element in Jahn (2011, Def. 5.23). Moreover, under further assumptions on X and W, it is equivalent to a properly maximal element in other senses (see Boţ et al., 2009, Props. 2.4.17 and 2.4.18). A weakly maximal element of $f(A)$ with respect to W is defined as a maximal element of $f(A)$ with respect to the algebraic interior of W. If $\mathrm{int}W \neq \varnothing$, the algebraic interior of W coincides with its topological interior, and the set of weakly maximal elements of $f(A)$ is given by $f(O(D, \mathrm{int}W))$ (see Jahn, 2011, Lemma 1.32 and Def. 4.12). If, in addition, $f(A) - (\mathrm{int}W \cup \{0\})$ is convex, the set of weakly maximal elements is equal to $\bigcup\{f(O(D, u)) : u \in (\mathrm{int}W)^+\}$ by Corollary 1. Thus, the lower bound basically corresponds to properly maximal elements and the upper bound to weakly maximal elements. Theorem 3 can be interpreted as bounding the set of maximal elements by the sets of properly and weakly maximal elements based on the idea of linear scalarization.

When Z and ${}^+Z$ are nonempty, ${}^+Z$ is a preference cone, as can be seen from Lemma 1(iii). In this case, we can apply Theorem 3 and Corollary 1 to $W = {}^+Z$ to obtain the following result.

Corollary 2. *Let $D = (A, X, f)$ be components of a decision-making problem with vector outcomes. Suppose that $Z \subseteq X^*$ and its dual ${}^+Z \subseteq X$ are nonempty. Then*

$$\bigcup\{O(D, u) : u \in Z\} \subseteq O(D, {}^+Z).$$

Suppose that $f(A) - ({}^+Z \cup \{0\})$ is convex and that $\mathrm{int}(f(A) - ({}^+Z \cup \{0\}))$ is nonempty. Then

$$O(D, {}^+Z) \subseteq \bigcup\{O(D, u) : u \in (\mathrm{int}\,{}^+Z)^+\}.$$

Suppose further that ${}^+Z$ is open and that $Z = ({}^+Z)^+$. Then

$$\bigcup\{O(D,u) : u \in Z\} = O(D, {}^+Z). \tag{8}$$

The relationships in (7) and (8) can be interpreted as follows. Suppose that the decision maker has preferences described by W. Then by considering all utilities representing W and collecting all maximizers of these utilities, we can obtain the set of the decision maker's optimal choices. That is, in order to characterize the optimal choices of a decision maker with incomplete preferences W, we can use complete preferences induced by utility u in W^+. Alternatively, suppose that the decision maker has utility u, about which there is uncertainty. If the modeler knows only that u belongs to some set $Z \subseteq X^*$, she can encompass all possible optimal choices by considering an imaginary decision maker who has incomplete preferences ${}^+Z$. On the other hand, if the decision maker is unsure about his own utility and is ambiguity averse in the sense that he chooses alternative a over status quo b only when $f(a)$ is better than $f(b)$ with respect to all utilities in Z, then his choice can be described as if he has incomplete preferences ${}^+Z$. We can think of another interpretation where there are multiple agents having utilities and the set of their utilities is given by Z. Suppose that they make a collective choice using the unanimity rule, that is, they choose alternative a over status quo b only when everyone prefers $f(a)$ to $f(b)$. Then the possible resulting outcomes in this scenario can be obtained by considering an "aggregate" agent who has incomplete preferences ${}^+Z$.

4 Games with Vector Outcomes

In this section, we study games with vector outcomes, building on the results derived in the previous section. A *game with vector outcomes* is defined by a tuple $(I, (A_i), (X_i), (f_i), (W_i))$, where I is a finite set, and for each $i \in I$, A_i is a nonempty set, X_i is a real topological vector space, f_i is a function from $A := \prod_{i\in I} A_i$ to X_i, and W_i is a preference cone in X_i. The set I denotes the set of players. For each player $i \in I$, A_i is the set of actions available to player i. We denote an action of player i by $a_i \in A_i$ and an action profile by $a = (a_1, \ldots, a_n) \in A$. We sometimes write $a = (a_i, a_{-i})$, where $a_{-i} \in A_{-i} := \prod_{j\in I\setminus\{i\}} A_j$. An action profile determines an outcome for each player. As in games with scalar payoffs, we allow that outcomes (or payoffs) differ across players. For each player $i \in I$, X_i is the outcome space for player i, and f_i is the function that assigns an outcome for player i to each action profile. That is, when an action profile a is chosen, player i receives outcome $f_i(a) \in X_i$. For each player $i \in I$, $W_i \subseteq X_i$ is the preference cone that describes player i's strict preferences over outcomes. For simplicity, we sometimes denote a game by (G, W), where $G = (I, (A_i), (X_i), (f_i))$ and $W = (W_i)$.

We define an equilibrium of a game with vector outcomes as an action profile from which no player has a profitable unilateral deviation.[21] Let $E(G, W)$ be the set of equilibria of the game (G, W), i.e.,

$$E(G, W) = \{a^* \in A : \nexists i \in I \text{ and } a_i \in A_i \text{ s.t. } f_i(a_i, a^*_{-i}) - f_i(a^*) \in W_i\}.$$

A game where each player i has a utility $u_i \in X_i^*$ is denoted by $(I, (A_i), (X_i), (f_i), (u_i))$, or simply by (G, u) where $u = (u_i)$. The game (G, u) can be considered as a standard strategic game $(I, (A_i), (v_i))$, where each player i has a scalar payoff function $v_i = u_i \circ f_i$ defined on the set of action profiles. Let $E(G, u)$ be the set of (Nash) equilibria of the game (G, u), i.e.,

$$E(G, u) = \{a^* \in A : \nexists i \in I \text{ and } a_i \in A_i \text{ s.t. } u_i(f_i(a_i, a^*_{-i})) > u_i(f_i(a^*))\}.$$

In the following theorem, we investigate the relationship between $E(G, W)$ and $E(G, u)$. We use the notation $F_i(a_{-i}) = \{f_i(a_i, a_{-i}) : a_i \in A_i\}$ for any $i \in I$ and $a_{-i} \in A_{-i}$.

Theorem 4. *Let (G, W) be a game with vector outcomes. Then*

$$\bigcup \{E(G, u) : u_i \in W_i^+ \ \forall i \in I\} \subseteq E(G, W).$$

Suppose, for any $i \in I$ and $a_{-i} \in A_{-i}$, that $F_i(a_{-i}) - (W_i \cup \{0\})$ is convex[22] *and that $\mathrm{int}(F_i(a_{-i}) - (W_i \cup \{0\}))$ is nonempty. Then*

$$E(G, W) \subseteq \bigcup \{E(G, u) : u_i \in (\mathrm{int} W_i)^+ \ \forall i \in I\}.$$

Proof. Let $a^* \in E(G, u)$ for some $u \in \prod_{i \in I} W_i^+$. Suppose to the contrary that $a^* \notin E(G, W)$. Then there exists $i \in I$ and $a_i \in A_i$ such that $f_i(a_i, a^*_{-i}) - f_i(a^*) \in W_i$. Since $u_i \in W_i^+$, we have $u_i(f_i(a_i, a^*_{-i}) - f_i(a^*)) > 0$ and thus $u_i(f_i(a_i, a^*_{-i})) > u_i(f_i(a^*))$. This contradicts $a^* \in E(G, u)$.

Suppose, for any $i \in I$ and $a_{-i} \in A_{-i}$, that $F_i(a_{-i}) - (W_i \cup \{0\})$ is convex and that $\mathrm{int}(F_i(a_{-i}) - (W_i \cup \{0\}))$ is nonempty. Choose any $a^* \in E(G, W)$. For every $i \in I$, we can obtain $u_i \in (\mathrm{int} W_i)^+$ such that $u_i(f_i(a^*)) \geq u_i(x)$ for all $x \in F_i(a^*_{-i})$, following the proof of Theorem 3. Hence, there exists $u \in \prod_{i \in I} (\mathrm{int} W_i)^+$ such that $a^* \in E(G, u)$. □

Theorem 4 extends Theorem 3 to game situations, offering a lower and upper bound

[21] This notion of equilibrium is a straightforward extension of the concept of Nash equilibrium to the context of games with vector payoffs, and so Bade (2005) calls it a Nash equilibrium. It is also called a Pareto equilibrium (Borm et al., 1988) and a Shapley equilibrium (Hamel and Löhne, 2018) in the literature.

[22] As mentioned in footnote 19, a sufficient condition for $F_i(a_{-i}) - (W_i \cup \{0\})$ to be convex is that A_i is a convex subset of a real vector space and f_i is W_i-concave in a_i for any a_{-i}. This sufficient condition is often assumed in the literature. See, for example, Bade (2005, Theorem 2) and Mármol et al. (2017, Theorem 2.5).

for the set of equilibria. If we adopt the terminologies used in the vector optimization literature, any Nash equilibrium of (G, u) where $u_i \in W_i^+$ for every i may be called a proper equilibrium of (G, W) and any Nash equilibrium of (G, u) where $u_i \in (\text{int} W_i)^+$ for every i a weak equilibrium of (G, W). Theorem 4 also generalizes Theorem 2 of Bade (2005), where she considers $X_i = X_i^* = \mathbb{R}^{m_i}$ and $W_i = \{x \in X_i : x \geq 0\} \setminus \{0\}$ for every i. In this scenario, we have $W_i^+ = \{u \in X_i^* : u \gg 0\}$ and $(\text{int} W_i)^+ = \{u \in X_i^* : u \geq 0\} \setminus \{0\}$.

The following corollary is an analogue of Corollary 1 in the context of games.

Corollary 3. *Let (G, W) be a game with vector outcomes. Suppose, for any $i \in I$ and $a_{-i} \in A_{-i}$, that W_i is open and that $F_i(a_{-i}) - (W_i \cup \{0\})$ is convex. Then*

$$E(G, W) = \bigcup \{E(G, u) : u_i \in W_i^+ \ \forall i \in I\}. \tag{9}$$

Corollary 3 generalizes Shapley's (1959) characterization of weak equilibria as well as that of Mármol et al. (2017, Theorem 2.5). From this result, we can see that the relationship $W_s = (W_w)^+$ drives these characterizations. While Corollary 3 does not cover the case of strong equilibria, Aumann (1962, Theorem C) generalizes Shapley's (1959) characterization of strong equilibria. As discussed after Corollary 1, the relationship in (9) holds if $F_i(a_{-i})$ is a convex polyhedron in a finite-dimensional vector space for all i and a_{-i}. On the other hand, Theorem 3 of Bade (2005) shows that $E(G, W) = \bigcup \{E(G, u) : u_i \in (\text{int} W_i)^+ \ \forall i \in I\}$ if every component of each f_i is strictly concave in a_i in the setting considered in Theorem 2 of Bade (2005).

We can write down an analogue of Corollary 2 as follows.

Corollary 4. *Let $G = (I, (A_i), (X_i), (f_i))$ be components of a game with vector outcomes. Suppose, for any $i \in I$, that $Z_i \subseteq X_i^*$ and its dual ${}^+Z_i \subseteq X_i$ are nonempty. Then*

$$\bigcup \{E(G, u) : u_i \in Z_i \ \forall i \in I\} \subseteq E(G, ({}^+Z_i)).$$

Suppose, for any $i \in I$ and $a_{-i} \in A_{-i}$, that $F_i(a_{-i}) - ({}^+Z_i \cup \{0\})$ is convex and that $\text{int}(F_i(a_{-i}) - ({}^+Z_i \cup \{0\}))$ *is nonempty. Then*

$$E(G, ({}^+Z_i)) \subseteq \bigcup \{E(G, u) : u_i \in (\text{int}\, {}^+Z_i)^+ \ \forall i \in I\}.$$

Suppose further, for any $i \in I$, that ${}^+Z_i$ is open and that $Z_i = ({}^+Z_i)^+$. Then

$$\bigcup \{E(G, u) : u_i \in Z_i \ \forall i \in I\} = E(G, ({}^+Z_i)). \tag{10}$$

The relationships in (9) and (10) can be interpreted as before. We can describe equilibria of a game with vector outcomes using Nash equilibria of scalarized games. If players have utilities, but they or the modeler is uncertain about their utilities, a conservative approach would be to consider players with incomplete preferences. Theorem 3.3 of Mármol et al. (2017) establishes the relationship in (10), focusing on the case where each Z_i is a polyhedral cone minus the origin in a Euclidean space.

An agent with incomplete preferences may not be able to compare many outcomes, and this often leads to a huge set of equilibria of a game with vector outcomes (see Bade, 2005). This large multiplicity of equilibria can also be seen from the characterization in Theorem 4. Typically, the dual of a preference cone contains infinitely many (normalized) utilities when it is nonempty. Hence, in order to obtain the lower or upper bound for the equilibrium set in Theorem 4, we need to find the Nash equilibria of infinitely many scalarized games (see Corley, 1985, Sec. 4, for a related remark). Given the large multiplicity of equilibria, it is important to predict which equilibrium players will actually play, or prescribe which one they should play. Based on the lower bound result in Theorem 4, we can take the following approach to address this issue. Suppose that a utility u_i representing player i's preference cone W_i is selected for every i. Then a Nash equilibrium of the scalarized game (G, u) gives an equilibrium of the game (G, W) with vector outcomes, while it has much smaller multiplicity. In particular, it is well-known that a generic finite strategic game with scalar payoffs has a finite number of mixed strategy Nash equilibria (see, for example, Wilson, 1971).[23] Hence, by focusing on a particular scalarization, we can narrow down the equilibrium set significantly.

The remaining question is which scalarization players will or should select. Although we do not aim to provide a rigorous theory on it in this paper, it can be argued that players are more likely to select a scalarization that renders the scalarized game natural or desirable properties (similarly to a focal point in Schelling, 1960). For example, suppose that the game (G, W) with vector outcomes is a two-player zero-sum game where $I = \{1, 2\}$, $X_1 = X_2 = \mathbb{R}^m$, $W_1 = W_2$, and $f_1 = -f_2$. Given that the game is zero-sum, it is natural that the players select a scalarization that makes the scalarized game zero-sum as well, which is achieved when they choose $u_1 = u_2$. Moreover, zero-sum games with scalar payoffs are known to possess many desirable properties such as interchangeability and best security levels of Nash equilibria, while such properties fail to hold in zero-sum games with vector payoffs (see Corley, 1985). So it can be argued that a symmetric scalarization (i.e., $u_1 = u_2$) should be selected for zero-sum games with vector outcomes. We illustrate this approach with the following example.

Example 1 (Example 3.1 of Corley, 1985)**.** Consider a two-player zero-sum game with

[23]As mentioned in the Introduction, we can deal with mixed strategies over a finite number of pure strategies and mixed strategy equilibria in our framework by interpreting A_i as the set of player i's mixed strategies and $f_i(a)$ as the expected vector outcome given the mixed strategy profile a.

vector outcomes where each player has two pure strategies, $X_1 = X_2 = \mathbb{R}^2$, $W_1 = W_2 = \{x \in \mathbb{R}^2 : x \geq 0\} \setminus \{0\}$, and f_1 on pure strategy profiles is given by

$$f_1 = \begin{bmatrix} (0,0) & (2,-1) \\ (1,-2) & (0,0) \end{bmatrix}.$$

Let us denote player 1's mixed strategy by $(p, 1-p)$ where $0 \leq p \leq 1$ and player 2's mixed strategy by $(q, 1-q)$ where $0 \leq q \leq 1$. Corley (1985) shows that the set of equilibria of this game is given by

$$\begin{gathered} \{(p,q) : p \in [0,1/3) \cup (2/3,1], q \in [0,1/3) \cup (2/3,1]\} \\ \cup \{(p,q) : 1/3 < p < 2/3, q = 0\} \cup \{(p,q) : p = 1, 1/3 < q < 2/3\}. \end{gathered}$$

The set of equilibria obtained from symmetric scalarizations is given by

$$\{(p,q) : p \in [0,1/3) \cup (2/3,1], q = 1-p\} \cup \{(p,q) : p = q = 1\},$$

which is a lot smaller than the set of equilibria of the original game.

Even when the game (G, W) with vector outcomes is non-zero-sum, there may exist utilities (u_i) that make the scalarized game (G, u) a zero-sum game. If players regard the properties of zero-sum games with scalar payoffs as important, they may coordinate on a scalarization that induces a zero-sum game. Consider a game (G, W) with vector outcomes where there are two players and each player i has k_i pure strategies, $\mathbb{R}^{m_i}$ as his outcome space, and the Pareto order as his preferences. Then utilities (u_1, u_2) inducing a zero-sum game with scalar payoffs can be characterized by a strictly positive solution to a homogeneous system of $k_1 \times k_2$ linear equations with $m_1 + m_2$ unknowns. Note that there may not exist such utilities. Even when scalarization cannot produce a zero-sum game, players can look for a scalarization that induces a game where each player's maximin strategy is his Nash equilibrium strategy. Pruzhansky (2011, Prop. 1) presents a condition for a game with scalar payoffs to have such a property, which is utilized in the next example.

Example 2. Consider a two-player game with vector outcomes where each player has two pure strategies, $X_1 = X_2 = \mathbb{R}^2$, $W_1 = W_2 = \{x \in \mathbb{R}^2 : x \geq 0\} \setminus \{0\}$, and f_1 and f_2 on pure strategy profiles are given by

$$f_1 = \begin{bmatrix} (1,0) & (-2,1) \\ (2,-1) & (0,1) \end{bmatrix} \quad \text{and} \quad f_2 = \begin{bmatrix} (1,-1) & (1,0) \\ (0,3) & (0,1) \end{bmatrix}.$$

Let us denote player 1's mixed strategy by $(p, 1-p)$ where $0 \leq p \leq 1$ and player 2's mixed strategy by $(q, 1-q)$ where $0 \leq q \leq 1$. Based on the characterization of Corley (1985, Theorem 2.1), we can obtain the set of equilibria of this game as follows:

$$\{(p,q) : p = 2/3, 0 < q < 1\} \cup \{(p,q) : 0 \leq p \leq 2/3, q = 1\}.$$

It can be checked that the scalarized game (G, u) is zero-sum if and only if $u_1 = u_2 = 0$; thus, there does not exist any utility pair $(u_1, u_2) \gg 0$ that makes (G, u) zero-sum. Using Proposition 1 of Pruzhansky (2011), we can show that the scalarized game (G, u) has a Nash equilibrium in which both players choose their maximin strategies if and only if u_1 is a positive multiple of $(1, 2)$ and u_2 is a positive multiple of $(3, 1)$. With such utilities (u_1, u_2), $(p, q) = (2/3, 2/3)$ is the unique Nash equilibrium of (G, u).

5 Conclusion

In this paper, we developed a general framework to study decision making and games with vector outcomes and presented some fundamental results. We started from an outcome space that is a real topological vector space. Our assumptions about preferences over outcomes allowed us to use a preference cone—defined as a nonempty convex cone satisfying a continuity axiom—to describe preferences. We defined a notion of utility representation and introduced a duality between outcomes and utilities. We presented conditions under which a preference cone admits a utility representation and is the dual of a set of utilities. We provided a lower and upper bound on the set of optimal choices in a decision-making problem with vector outcomes by using optimal choices in related decision-making problems where payoffs are scalarized by utilities representing the decision maker's preferences. Similarly, we characterized the set of equilibria of a game with vector outcomes by Nash equilibria of related games where payoffs are scalarized by utilities representing the players' preferences.

In many real-world scenarios, outcomes are vectors and agents have incomplete preferences over outcomes. Our framework can be used to model and analyze such scenarios. While most existing work on games with vector payoffs assumes that an outcome space is finite-dimensional Euclidean space and that preferences are given by the Pareto order, our framework allows infinite-dimensional outcome spaces and preferences described by a preference cone, and thus it has broader applicability. As games with vector payoffs have not been much explored yet in the existing literature, we regard our work as one of early studies on this topic, and there are many questions that remain to be investigated in future work. Our focus in this paper is on noncooperative simultaneous-move games with complete information, but other game models such as sequential-move games, games with incomplete information, and cooperative games can be examined with vector outcomes.

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