# Preemptive Entry in Sequential Auctions with Participation Cost*

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**Abstract**

This paper analyzes a scenario in which two objects are sold in sequence at two second-price auctions. There are two bidders, and each bidder's valuations of the two objects are affiliated. Participating in each auction is costly. Bidders decide whether to enter each auction, observing their entry decisions in any previous auction. We study the properties of equilibria and provide a sufficient condition for their existence. Due to affiliation, a bidder's entering the first auction may signal his strong interest in the second object. Hence, a bidder with a higher valuation of the second object tends to participate in the first auction more aggressively in order to preempt the opponent's entry into the second auction. Because of this signaling motive, the sequential auction format can generate higher revenue in the first auction and lower revenue in the second auction than those obtained by the simultaneous counterpart.

**Keywords:** Sequential auctions, participation cost, preemptive entry, signaling.
**JEL Classification:** D44, D82.

## 1 Introduction

Consider a scenario where related items (for example, paintings, antiques, and wines) are put up for sale at sequential auctions and potential bidders face participation cost in each

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auction. Participating and bidding in an auction is a costly activity for a bidder, as it takes time and effort to design a bidding strategy and execute it.[1] Since the items are related, it is natural that a bidder's valuations of the items are positively correlated. Hence, a bidder's active participation in earlier auctions may signal his high valuations of the later items. Then, after observing a bidder's active participation in earlier auctions, other bidders may decide not to participate any more.[2] In other words, the presence of participation cost in sequential auctions of related items creates the possibility of signaling and entry deterrence. So a bidder who is eager to acquire an item sold at a later auction may benefit from participating aggressively in earlier auctions. The objective of this paper is to present an auction model in which bidders have an incentive for such signaling and entry deterrence and study its implications on bidders' equilibrium behavior and auction performance.

For analytical tractability, we consider a simple setting with two objects and two potential bidders. The two bidders have independently and identically distributed private valuations of the two objects. Each bidder's valuations of the two objects are affiliated, so that if a bidder has a high valuation of one object, his valuation of the other object is likely to be high.[3] The two objects are sold at two second-price auctions held sequentially, one object at a time. In each auction, bidders simultaneously choose whether to participate, and in case a bidder participates, he submits a bid, incurring a cost. To simplify our analysis, we assume that bidders learn only their entry decisions in the first auction before the second auction begins. Under this assumption, it is optimal for a participating bidder to bid his valuation of the object not only in the second auction but also in the first one. Then to describe a bidder's strategy, it suffices to specify his entry cutoffs for the two auctions, so that a bidder enters an auction if and only if his valuation of the object exceeds the cutoff for the auction.

Since a bidder participates in the first auction only when his valuation of the first object exceeds his cutoff, a bidder's entering the first auction signals his strength in the second auction due to the affiliation assumption. In other words, a bidder who enters the first auction is stronger in the second auction than one who does not enter in the sense that the distribution of a bidder's valuation of the second object conditional on that he enters first-order stochastically dominates that conditional on that he does not enter. As a result, by entering the first auction, a bidder can induce his opponent to adopt a higher cutoff in the second auction, regardless of whether his opponent entered the first auction or not, and this results in gains in his expected payoff from the second auction. Moreover, these gains

[1] A positive entry fee can also bring about a cost of participation.

[2] Von der Fehr (1994) points out that a bidder present in an auction house will not participate in an auction actively if he believes his chance of winning at a price below his willingness to pay is small.

[3] In the auction literature, affiliation is usually assumed on different bidders' signals in interdependent values models (see, for example, Milgrom and Weber, 1982). In contrast, we impose affiliation on a single bidder's valuations of different objects in an independent private values model.

become weakly larger as a bidder's valuation of the second object increases. In other words, a bidder who has a high valuation of the second object has a stronger incentive to signal his strength in the second auction, and thus he is more likely to enter the first auction. This results in a bidder's cutoff function for the first auction that is weakly decreasing in his valuation of the second object. Note that the decreasing property of a cutoff function reinforces the signaling effect. We characterize equilibrium cutoffs for the two auctions, and we present a sufficient condition to prove the existence of equilibria applying the Schauder fixed point theorem.

A natural benchmark to the sequential auction format analyzed in this paper is the scenario where the two second-price auctions are held simultaneously. In the simultaneous auction format, bidders obtain no information that would allow them to update their beliefs about their opponents' valuations, and a strategy is represented by two cutoffs for the two auctions. We show that, in each auction, the equilibrium cutoff in the simultaneous auction format is bounded by the corresponding ones in the sequential auction format. In the sequential auction format, bidders become strong or weak in the second auction depending on their entry decisions in the first auction. In the simultaneous auction format, in contrast, no such inference is made. Thus, the equilibrium cutoffs for the second auction following different histories in the sequential auction format are dispersed around the equilibrium cutoff for the second auction in the simultaneous auction format. Due to the gains in the second auction from entering the first auction in the sequential auction format, a bidder with a high valuation of the second object participates in the first auction more aggressively in the sequential auction format than in the simultaneous auction format, and this induces a bidder with a low valuation of the second object to participate in the first auction less aggressively in the sequential auction format. Thus, the equilibrium cutoff for the first auction in the simultaneous auction format is between the maximum and minimum values of the equilibrium cutoff function for the first auction in the sequential auction format.

We evaluate the performance of an auction by the four measures of the expected revenue, the expected price conditional on that the object is sold, the expected bidder surplus, and the expected social surplus. Since equilibrium cutoffs in the two auctions are entangled in a complex way, it is difficult to calculate equilibrium cutoffs and compare the performance measures in the sequential and simultaneous auction formats analytically. So in order to simplify the calculation and make the comparison possible, we consider a simple scenario where the two objects are identical providing an equal valuation (i.e., constant marginal valuations) and each bidder's valuation of the object is uniformly distributed on $[0, 1]$. With identical objects, cutoffs for the first auction can be represented by a number rather than a function. Due to the signaling and entry deterrence incentive, the equilibrium cutoff for the first auction is lower in the sequential auction format than in the simultaneous counterpart.

This means that competition in the first auction is more intense in the sequential auction format than in the simultaneous auction format, and thus the sequential auction format yields a higher expected revenue, a higher expected price, a lower expected bidder surplus, and a lower expected social surplus in the first auction. On the other hand, the sequential auction format enables bidders to achieve coordination in the second auction using the history in the first auction, and this mitigates competition and saves participation cost in the second auction. Hence, the effects of the sequential auction format on the four measures in the second auction work in the opposite direction to those in the first auction. Overall, the effects on the second auction dominate those on the first auction, and thus the sequential auction format improves the expected bidder surplus and the expected social surplus while it reduces the expected revenue and the expected price summed over the two auctions. The effects on the first auction are limited because the impact of more aggressive participation by a bidder with a high valuation of the second object is partially offset by that of less aggressive participation by one with a low valuation. The result that the expected price of the first object is higher than that of the second one can shed light on the widely-observed pattern of declining price sequences, which is known as the declining price anomaly (see, for example, Ashenfelter, 1989). When bidders have a signaling and entry deterrence motive, they participate in earlier auctions more aggressively, and this provides an explanation for the declining price anomaly. This result also suggests that, when the objects are owned by different sellers, sellers will prefer to have their objects put up for sale earlier, and there will be competition among sellers to become the first seller.

This paper is related to the literature on auctions with costly participation. Lee and Park (2016) briefly review existing studies in this literature. Among them, a closely related paper to ours is Tan and Yilankaya (2006), who study a second-price auction with symmetric and asymmetric bidders. With slight adjustment, their analysis can be applied to analyze the second auction in our model, where bidders can be symmetric or asymmetric depending on their entry decisions in the first auction. As noted by Lee and Park (2016), most existing studies in this literature assume that bidders make entry decisions simultaneously, and thus they cannot capture any preemptive motive. In contrast, McAdams (2015) considers a second-price auction with multiple bidding rounds. Bidders with higher valuations submit earlier bids in equilibrium, as they have stronger incentives to deter other bidders' participation. Lee and Park (2016) analyze a second-price auction where bidders make entry decisions sequentially in an exogenous order. Due to participation costs, a later bidder's equilibrium entry cutoff exceeds that of any earlier participating bidder. So earlier bidders have a preemptive advantage, and thus they are more likely to participate than later bidders. Both McAdams (2015) and Lee and Park (2016) consider an auction of a single object, while in this paper we study two auctions held sequentially to sell two objects. By analyzing sequential auctions of two objects, we can address the preemptive effect of entry across auctions, instead of that within an auction.

Since we examine sequential auctions of two objects, this paper also fits into the literature on multiple-object sequential auctions. Milgrom and Weber (2000) study various forms of auctions, including sequential auctions, to sell multiple units of an identical object. Assuming single-unit demand, they analyze equilibrium behavior, price sequences, and revenue comparison. Weber (1983) discusses the case of multi-unit demand and auctions of non-identical objects as well. Ortega-Reichert (1968) is an early work that studies a signaling incentive in sequential auctions. He considers sequential first-price auctions of two objects with two bidders in an independent private values setting. In his model, a bidder has an incentive to bid low in the first auction in order to induce his opponent to bid low in the second auction. So the signaling incentive works in the opposite direction to that in our model.[4] Katzman (1999) studies sequential second-price auctions of two units of an identical object, where a bidder has multi-unit demand with diminishing marginal valuations. A bidder's winning in the first auction reduces his valuation in the second auction, and thus the winning bidder in the first auction becomes relatively weaker in the second auction. So there exists a symmetric equilibrium in which a bidder shades his bid in the first auction below his valuation of the first unit. Lamy (2012) considers a generalization of Katzman's (1999) model in terms of the distribution of valuations and characterizes symmetric equilibria. Recently, Kong (2017) studies sequential auctions of two objects with synergy and affiliation and performs a structural analysis. Due to synergy, a bidder's winning the first object increases his valuation of the second object, and this induces bidders to bid high in the first auction in equilibrium compared to the case of no synergy.

This paper lies at the intersection of the above two literatures, and there are a few other papers at this intersection. Von der Fehr (1994) considers sequential English auctions of two units of an identical object where bidders have single-unit demand and face participation costs. The highest bidding loser in the first auction wins the object at a low price in the second auction, because some bidders will drop out from the second auction given participation costs. Due to the information revelation in the first auction, bidders have an incentive for predatory bidding in the first auction. Menezes and Monteiro (1997) examine sequential second-price auctions of two objects where bidders can demand both objects. They assume that a bidder learns his valuation of the second object only after the first auction and that, if a bidder does not enter the first auction, he cannot participate in the

[4] In Ortega-Reichert (1968), a bidder's valuations of two objects are positively correlated, as in our model. If they are negatively correlated in our model, the signaling incentive will work in the opposite direction: that is, a bidder with a high valuation of the second object will be more careful about entering the first auction than one with a low valuation, as entering the first auction now signals his weakness in the second auction. This observation suggests that the assumption of affiliation plays a key role in deriving our results.

second auction. With stochastically independent objects, they analyze bidders' equilibrium behavior. This paper also touches on the literature on preemptive bidding and signaling in auctions. For example, Fishman (1988) studies a takeover bidding process in which two bidders bid sequentially and shows that in equilibrium the first bidder may make a preemptive high initial bid in order to signal his high valuation and deter the second bidder from competing. Avery (1998) provides a rationale for jump bidding in an English auction in order to signal one's aggressive strategy. In comparison with these papers, our analysis reveals that entry can play a signaling role and thus preemptive signaling behavior can exist even in a scenario where bids are not disclosed.

The rest of this paper is organized as follows. We describe the sequential auction model in Section 2, and we analyze bidders' equilibrium behavior in Section 3. We compare the sequential auction format and the simultaneous counterpart in terms of equilibrium cutoffs and the four performance measures in Section 4, and we conclude and discuss possible extensions in Section 5. All the proofs are relegated to an Appendix.

## 2 Model

A seller has two indivisible objects, called objects $A$ and $B$. There are two potential bidders, called bidders 1 and 2. Bidder $i$ has valuation $v_i^k \in [0,1]$ of object $k$, for $i = 1,2$ and $k = A,B$. The two bidders' valuations $(v_1^A, v_1^B)$ and $(v_2^A, v_2^B)$ are identically and independently distributed following a probability density function $f : [0,1]^2 \to \mathbb{R}$. We assume that the joint density function $f$ has full support on $[0,1]^2$ and is continuous on its support.[5] We assume that for each bidder $i$, the two valuations $v_i^A$ and $v_i^B$ are *affiliated* in a strong sense, i.e., for any $0 < x < x' < 1$ and $0 < y < y' < 1$,[6]

$$f(x,y)f(x',y') > f(x',y)f(x,y'). \tag{1}$$

Roughly speaking, our affiliation assumption means that if a bidder's valuation of object $A$ is high, then his valuation of object $B$ is likely to be high as well, and vice versa. When there is no need to specify a bidder, we will use $v^k$ to denote a valuation of object $k$, for $k = A,B$. We use $f_k$ and $F_k$ to denote the marginal density function and the marginal cumulative distribution function, respectively, of $v^k$, for $k = A,B$.

[5]Our analysis can be extended to the case where the support of $f$ is a nicely-shaped proper subset of $[0,1]^2$. However, this requires more notations and the results get more complicated without adding much insight. So we focus on the case of full support for analytic convenience.

[6]In fact, for our analysis, it suffices to have the inequality in (1) hold weakly and it hold strictly for all $(x,y)$ near $(0,0)$ and all $(x',y')$ near $(1,1)$ in the following sense: for any $\varepsilon > 0$, there exist an open ball $B_1 \in [0,1]^2$ in the $\varepsilon$-neighborhood of $(0,0)$ and another open ball $B_2 \in [0,1]^2$ in the $\varepsilon$-neighborhood of $(1,1)$ such that $f(x,y)f(x',y') > f(x',y)f(x,y')$ for all $(x,y) \in B_1$ and all $(x',y') \in B_2$. Refer to Eq. (17).

As an example that satisfies our assumptions on the distribution, consider the Clayton copula whose joint cumulative distribution function is given by

$$F(x,y) = (x^{-\theta} + y^{-\theta} - 1)^{-1/\theta} \tag{2}$$

for $(x,y) \in [0,1]^2$, where $\theta > 0$. The support of the Clayton copula is $[0,1]^2$, and the joint density $f$ of the distribution satisfies the strict affiliation inequality (1) because

$$\frac{\partial^2}{\partial x \partial y} \ln f(x,y) = \theta(1+2\theta) \frac{x^{-\theta-1} y^{-\theta-1}}{(x^{-\theta} + y^{-\theta} - 1)^2} > 0$$

for all $(x,y) \in (0,1)^2$ (see Krishna, 2009, p. 285).

The seller holds two sealed-bid second-price auctions sequentially, first to sell object $A$ and then to sell object $B$. We refer to the auction to sell object $k$ as auction $k$, for $k = A, B$. Participating in an auction is costly, and a bidder incurs cost $c^k \in (0,1)$ when he enters auction $k = A, B$. The seller's valuation of object $k = A, B$ is denoted by $r^k \in [0,1]$. We assume that $c^k + r^k < 1$ for $k = A, B$. The seller sets a reserve price for object $k = A, B$ equal to her valuation $r^k$ in order to avoid any object being sold at a price below her valuation, and the reserve prices are known to the bidders. Let $p_i^k$ denote bidder $i$'s payment in auction $k$ if he wins, which can be the other bidder's bid or the reserve price. Bidder $i$'s payoff in auction $k$ is given by $v_i^k - p_i^k - c^k$ if he participates and wins, $-c^k$ if he participates and does not win, and 0 if he does not participate. A bidder's total payoff is the sum of his payoffs in the two auctions.[7] Each bidder maximizes his expected payoff given his information. Before auction $A$ begins, each bidder learns his own valuations of the two objects but not the other's. In each auction, each bidder decides whether to participate or not and makes a bid in case he participates. After auction $A$ and before auction $B$, the seller reveals the bidders' entry decisions in auction $A$, but not their bids. After auction $B$, the winning bidders and their payments in the two auctions are announced.

# 3 Equilibrium Characterization and Existence

In this section, we analyze bidders' equilibrium behavior in the sequential auction model described in Section 2. Provided that a bidder enters auction $k$, it is weakly dominant for the bidder to bid his own valuation of object $k$. Hence, we focus on strategies in which a bidder bids his own valuation when he enters an auction. Then the remaining things to specify in a strategy are a bidder's entry decision rules in the two auctions. Using a standard envelope theorem argument, we can see that bidder $i$'s expected payoff from participating

[7] Note that bidders are not subject to budget constraints in our model. For an analysis of multiple-object sequential auctions with budget constrained bidders, see, for example, Benoît and Krishna (2001).

in auction $k$ is nondecreasing in $v_i^k$. Thus, it is natural to focus on cutoff strategies in which a bidder enters auction $k$ if and only if his valuation of object $k$ exceeds some cutoff given his information. There are four possible histories after auction $A$, and given a history, $v_i^A$ does not affect bidder $i$'s payoff in auction $B$. So a bidder's cutoff for auction $B$ depends on the history but not on his valuation of object $A$. In contrast, when making an entry decision in auction $A$, a bidder considers not only his payoff in auction $A$ but also that in auction $B$, and thus his cutoff for auction $A$ may depend on his valuation of object $B$. Hence, we can represent a cutoff strategy by

$$(e, e^{oo}, e^{oi}, e^{io}, e^{ii})$$

where $e : [0,1] \to [0,1]$ and $e^{oo}, e^{oi}, e^{io}, e^{ii} \in [0,1]$. The function $e$ describes a bidder's entry cutoff for auction $A$ depending on his valuation of object $B$. That is, a bidder with valuations $(v^A, v^B)$ participates in auction $A$ if and only if $v^A > e(v^B)$. The real number $e^{oo}$ denotes a bidder's entry cutoff for auction $B$ after neither participates in auction $A$, $e^{oi}$ after the bidder does not participate in auction $A$ while the other does, $e^{io}$ after the bidder participates in auction $A$ while the other does not, and $e^{ii}$ after both participate in auction $A$.

We focus on a symmetric perfect Bayesian equilibrium in truthful bidding cutoff strategies and call it simply an equilibrium. In the following proposition, we study equilibrium cutoffs for auction $B$ given a cutoff function for auction $A$.

**Proposition 1.** *Suppose that the bidders use a cutoff function* $e : [0,1] \to [0,1]$ *for auction* $A$ *such that* $e$ *is nonincreasing on* $[0,1]$*, is constant on* $[0, r^B + c^B]$ *with* $e(0) > 0$*, and satisfies* $e(v^B) < 1$ *on some non-degenerate interval. Then there exist equilibrium cutoffs* $(e^{oo}, e^{oi}, e^{io}, e^{ii})$ *for auction* $B$ *such that* $e^{io} < e^{oi}$*, and any such equilibrium cutoffs satisfy* $r^B + c^B \leq e^{io} \leq e^{oo} < e^{ii} < e^{oi} \leq 1$*, with* $e^{io} < e^{oo}$ *if and only if* $e(v^B) > 0$ *for some* $v^B > r^B + c^B$*. Suppose in addition that* $e(0) < 1$*,* $f$ *is nonincreasing in the second argument, and* $\int_{e(y)}^{1} f(x,y)dx$ *is nonincreasing in* $y$*. Then equilibrium cutoffs for auction* $B$ *are uniquely determined.*

Proposition 1 presents sufficient conditions for the existence and uniqueness of equilibrium cutoffs for auction $B$ as well as their properties. The idea of the proof is as follows. Given a cutoff function $e$ used by a bidder for auction $A$, we can derive the cumulative distribution functions of his valuation of object $B$ conditional on that he participates and does not participate in auction $A$, and call them $G(\cdot|in)$ and $G(\cdot|out)$, respectively. A bidder's participation in auction $A$ means that his valuation of object $A$ is high enough to exceed his cutoff, which suggests a high valuation of object $B$ by our assumption of affiliation. Furthermore, when $e$ is nonincreasing, entering auction $A$ is more likely when the bidder has a high valuation of object $B$. These two effects imply that $G(\cdot|in)$ first-order stochastically dominates $G(\cdot|out)$ in a strong sense (i.e., $G(z|in) < G(z|out)$ for all $z \in (0,1)$). Then, in the language of Tan and Yilankaya (2006, Sec. 4), entering auction $A$ makes a bidder "strong" in auction $B$, while not entering makes him "weak."

We can characterize equilibrium cutoffs for auction $B$ by considering the cases of symmetric and asymmetric bidders as in Tan and Yilankaya (2006).[8] After both bidders participate or neither participates in auction $A$, we have symmetric bidders with distribution $G(\cdot|in)$ or $G(\cdot|out)$. There is a unique symmetric equilibrium cutoff in each of these cases (cf. Proposition 1 of Tan and Yilankaya, 2006). After only one bidder participates in auction $A$, we have asymmetric bidders where the bidder participating in auction $A$ is strong and the other bidder is weak. We focus on "intuitive" equilibria, as named by Tan and Yilankaya (2006), in which the strong bidder is more likely to participate than the weak one (i.e., $e^{io} < e^{oi}$). As in Proposition 4 of Tan and Yilankaya (2006), we can establish the existence of intuitive equilibria, but they may not be unique.

The order of the equilibrium cutoffs, $e^{io} < e^{oo} < e^{ii} < e^{oi}$, is intuitive.[9] First, let us compare the two cases of symmetric bidders. When high valuations are more likely, bidders will be more cautious about entering, which leads to $e^{oo} < e^{ii}$. Also, when facing the same type of the other bidder, a strong bidder will be more aggressive than a weak bidder, which leads to $e^{io} < e^{oo}$ and $e^{ii} < e^{oi}$. The most optimistic scenario for a bidder participating in auction $B$ is that the opponent never participates, which gives him the payoff $v^B - r^B - c^B$. Hence, a bidder with valuation $v^B < r^B + c^B$ has no incentive to participate in auction $B$, and the equilibrium cutoffs should be at least as large as $r^B + c^B$.

As mentioned above, symmetric equilibrium cutoffs in the cases of symmetric bidders are uniquely determined, while intuitive equilibria in the cases of asymmetric bidders may not be unique. By Proposition 4 of Tan and Yilankaya (2006), we can establish the uniqueness of intuitive equilibria if $G(\cdot|in)$ and $G(\cdot|out)$ are both concave on $[0,1]$. The derivative of $G(\cdot|out)$ at $z$ is proportional to $\int_0^{e(z)} f(x,z)dx$. Since $e$ is nonincreasing, the assumption that $f$ is nonincreasing in the second argument guarantees that $\int_0^{e(z)} f(x,z)dx$ is nonincreasing in $z$, making $G(\cdot|out)$ concave on $[0,1]$. For $G(\cdot|in)$ to be concave on $[0,1]$, the support of $G(\cdot|in)$ needs to be $[0,1]$, which is implied by the assumption that $e(0) < 1$. The derivative of $G(\cdot|in)$ at $z$ is proportional to $\int_{e(z)}^1 f(x,z)dx$. The assumption that $\int_{e(y)}^1 f(x,y)dx$ is nonincreasing in $y$ implies that $G(\cdot|in)$ is concave on $[0,1]$. So the additional assumptions guarantee that there exists a unique intuitive equilibrium in the cases of asymmetric bidders.

[8] A difference with Tan and Yilankaya (2006) is that they assume that the distributions of valuations have full support on $[0,1]$ while in our analysis $G(\cdot|in)$ and $G(\cdot|out)$ have supports $[\underline{v}^B, 1]$ and $[0, \overline{v}^B]$, respectively, where $0 \leq \underline{v}^B \leq \overline{v}^B \leq 1$.

[9] While Proposition 1 shows that we have $e^{io} = e^{oo}$ when $e(v^B) = 0$ for all $v^B > r^B + c^B$, Proposition 2 proves that any equilibrium cutoff function $e$ for auction $A$ does not satisfy this property and thus $e^{io} < e^{oo}$ must hold in equilibrium.

Moreover, by Proposition 5 of Tan and Yilankaya (2006), the concavity of $G(\cdot|out)$ implies that there is no equilibrium other than the intuitive equilibrium.[10]

The concavity of a distribution function means that smaller values have higher probability density. So $f$ being nonincreasing in the second argument is necessary for both $G(\cdot|in)$ and $G(\cdot|out)$ to be concave on $[0,1]$ at the same time. Moreover, if $f$ is decreasing in the second argument and $e$ does not decrease too rapidly, $\int_{e(y)}^{1} f(x,y)dx$ will be nonincreasing in $y$. While the sufficient condition to obtain the uniqueness result in Proposition 1 is strong, that to obtain the existence result is mild. In the next proposition, we study the properties that any equilibrium cutoff function for auction $A$ should possess.

**Proposition 2.** *Suppose that $(e, e^{oo}, e^{oi}, e^{io}, e^{ii})$ is an equilibrium such that $e$ is continuous almost everywhere and $r^B + c^B \le e^{io} \le e^{oo} < e^{ii} < e^{oi} \le 1$. Then $e^{io} < e^{oo}$, $e$ is continuous and nonincreasing on $\{v^B \in [0,1] : e(v^B) > r^A\}$, it is constant on $[0, e^{io}]$ with $e(0) > r^A + c^A$ and decreasing on $[e^{io}, e^{oi}] \cap \{v^B \in [0,1] : r^A < e(v^B) < 1\}$, and it is continuously differentiable on $\{v^B \in [0,1] : r^A < e(v^B) < 1\}$ except at $e^{io}$, $e^{oo}$, $e^{ii}$, and $e^{oi}$. Moreover, $e$ belongs to one of the following three cases:*
*(i) $e(v^B) > r^A$ for all $v^B \in [0,1]$, and $e$ is constant on $[e^{oi}, 1]$,*
*(ii) There exists $\overline{v}^B \in (e^{oo}, e^{oi})$ such that $e(v^B) > r^A$ for all $v^B \in [0, \overline{v}^B)$, $\lim_{v^B \to (\overline{v}^B)^-} e(v^B) = r^A$, $0 \le e(\overline{v}^B) \le r^A$, and $e(v^B) = 0$ for all $v^B \in (\overline{v}^B, 1]$,*[11]
*(iii) $e(v^B) > r^A$ for all $v^B \in [0, e^{oi})$, $\lim_{v^B \to (e^{oi})^-} e(v^B) = r^A$, and $0 \le e(v^B) \le r^A$ for all $v^B \in [e^{oi}, 1]$.*
*In each case, if $e(v^B) = 1$ for some $v^B \in [0,1]$, then the set $\{v^B \in [0,1] : e(v^B) = 1\}$ is equal to $[0, \underline{v}^B]$ for some $\underline{v}^B \in [e^{io}, e^{ii})$.*

Proposition 2 describes the properties of an equilibrium cutoff function for auction $A$. Below we explain how to derive them. Consider a bidder with valuations $(v^A, v^B)$, and suppose that the other bidder uses a cutoff function $e$ for auction $A$ while the two bidders use equilibrium cutoffs $(e^{oo}, e^{oi}, e^{io}, e^{ii})$ for auction $B$ given $e$. Let $\pi^A(v^A)$ be the bidder's expected payoff in auction $A$ provided that he participates. If $v^A < r^A$, the bidder has no chance of winning object $A$ even when he participates in auction $A$. So we have $\pi^A(v^A) = -c^A$ for all $v^A < r^A$. As the bidder has a higher valuation of object $A$ in $[r^A, 1]$, he obtains a higher expected payoff from auction $A$, that is, $\pi^A(v^A)$ is increasing in $v^A$ on $[r^A, 1]$. Let $\Pi^B(v^B)$ be the bidder's payoff gain in auction $B$ from participating in auction $A$ (before learning the other bidder's entry decision in auction $A$). By entering auction $A$, the bidder can make himself stronger in auction $B$, which raises his opponent's cutoff for

[10]The concavity of $G(\cdot|in)$ and $G(\cdot|out)$ also implies that there is no asymmetric equilibrium in the cases of symmetric bidders (see Proposition 2 of Tan and Yilankaya, 2006). Hence, under the additional assumptions, there is a unique equilibrium in each case of symmetric bidders even if we do not focus on symmetric equilibria.

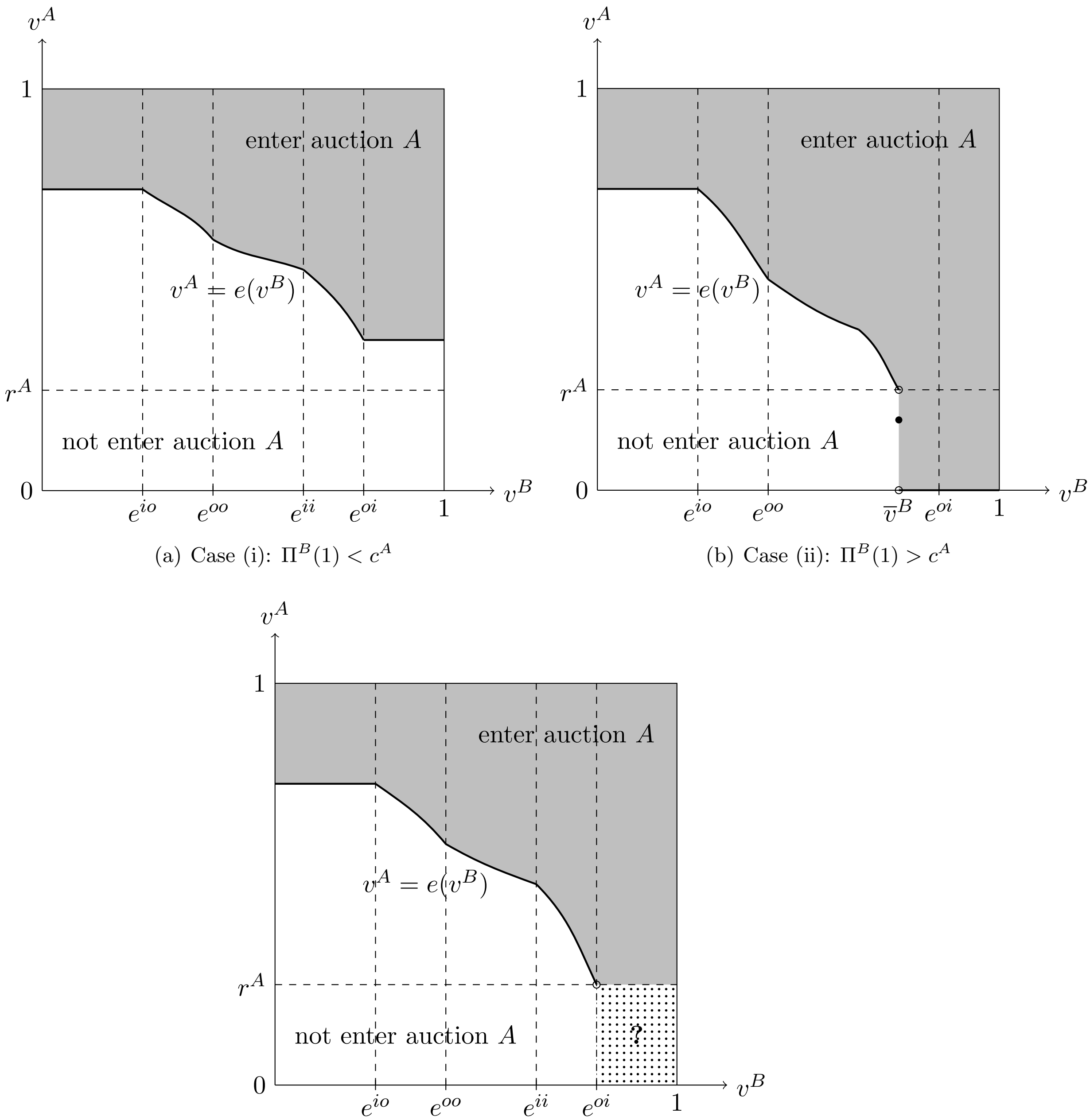
[11]More precisely, $e(v^B) = 0$ for almost every $v^B \in [\overline{v}^B, e^{ii}]$ if $\overline{v}^B \in (e^{oo}, e^{ii})$.

auction $B$. Facing a higher cutoff of the opponent has two beneficial effects. The bidder is more likely to win, and his payment can become lower. So participation in auction $A$ induces a higher expected payoff in auction $B$, that is, $\Pi^B(v^B) \geq 0$ for all $v^B \in [0,1]$. Moreover, the bidder's gain in auction $B$ weakly increases in his valuation of object $B$, that is, $\Pi^B(v^B)$ is nondecreasing in $v^B$ on $[0,1]$. When $v^B$ is very low (i.e., $v^B \leq e^{io}$), the bidder never participates in auction $B$ regardless of the entry decisions in auction $A$. Thus, there are no gains in auction $B$ from participating in auction $A$ for sufficiently low $v^B$. When $v^B$ is very high (i.e., $v^B \geq e^{oi}$), the bidder always participates in auction $B$. In this case, the probability of winning object $B$ is not affected by whether the bidder enters auction $A$ or not, but his participation in auction $A$ may reduce the payment. Hence, there are positive gains in auction $B$ from participating in auction $A$, but the magnitude is independent of $v^B$ for sufficiently high $v^B$. So we have constant $\Pi^B$ on the intervals $[0, e^{io}]$ and $[e^{oi}, 1]$ with $\Pi^B(0) = 0$ and $\Pi^B(1) > 0$.

When making the entry decision in auction $A$, the bidder compares $\pi^A(v^A) + \Pi^B(v^B)$ with 0. He prefers to enter auction $A$ if and only if $\pi^A(v^A) + \Pi^B(v^B) > 0$. So when the equilibrium cutoff $e(v^B)$ is in the interior of $[0,1]$, a bidder of the cutoff type should be indifferent between participating in auction $A$ and not participating, that is, $\pi^A(e(v^B)) + \Pi^B(v^B) = 0$. Since $\pi^A$ is increasing on $[r^A, 1]$, the shape of the equilibrium cutoff function $e$ reflects that of $\Pi^B$ in the region where $r^A < e(v^B) < 1$. In particular, in this region, $e$ is continuous, constant on $[0, e^{io}]$ and $[e^{oi}, 1]$, decreasing on $[e^{io}, e^{oi}]$, and continuously differentiable except at $e^{io}$, $e^{oo}$, $e^{ii}$, and $e^{oi}$.

By comparing $\Pi^B(1)$ and $c^A$, we can consider three cases, which are illustrated in Figure 1. The first case is where $\Pi^B(1) < c^A$. In this case, the participation cost for auction $A$ is not so small that only a bidder whose valuation of object $A$ exceeds the reserve price $r^A$ enters auction $A$. That is, we have $e(v^B) > r^A$ for all $v^B \in [0,1]$. The second case is where $\Pi^B(1) > c^A$. In this case, the participation cost for auction $A$ is so small that a bidder with a sufficiently high valuation of object $B$ finds it profitable to enter auction $A$ regardless of his valuation of object $A$. In other words, a bidder with a high $v^B$ obtains a large gain in auction $B$ from entering auction $A$, and so he is willing to enter auction $A$ even when he has no chance of winning object $A$. In this case, there is a threshold type $\overline{v}^B \in (e^{oo}, e^{oi})$ such that $e(v^B) > r^A$ for all $v^B < \overline{v}^B$ and $e(v^B) = 0$ for all $v^B > \overline{v}^B$. A bidder with $v^A < r^A$ and $v^B > \overline{v}^B$ participates in auction $A$ just for the signaling purpose without any intent to obtain object $A$.[12] Lastly, the third case is where $\Pi^B(1) = c^A$. Since $\Pi^B$ is constant on $[e^{oi}, 1]$ and $\pi^A(v^A) = -c^A$ for all $v^A \in [0, r^A]$, a bidder with valuations $(v^A, v^B) \in [0, r^A] \times [e^{oi}, 1]$ is indifferent between entering and not entering auction $A$. Hence,

[12]This feature is analogous to indicative bidding in that the first stage serves the purpose of signaling for the second stage. However, signaling is costly in our model, whereas indicative bidding is nonbinding and thus involves no cost. See Quint and Hendricks (2018) for an analysis of indicative bidding.




(a) Case (i): $\Pi^B(1) < c^A$

(b) Case (ii): $\Pi^B(1) > c^A$

(c) Case (iii): $\Pi^B(1) = c^A$

Figure 1: Shape of Equilibrium Cutoff Function $e$.

in this case, we cannot pin down the values of $e$ on $[e^{oi}, 1]$ from the indifference condition. In contrast, in the previous two cases, the indifference condition determines $e$ completely on the entire interval $[0, 1]$ (at least almost everywhere).

We can regard the last case as a degenerate case and the first two as non-degenerate. It can be checked that, in the first two cases, any equilibrium cutoff function $e$ for auction $A$ satisfies the sufficient condition for the existence result in Proposition 1. In the third case, $e$ may not be nonincreasing on $[e^{oi}, 1]$, but as long as the supports of the induced distributions $G(\cdot|in)$ and $G(\cdot|out)$ are intervals and $G(z|in) < G(z|out)$ holds on their intersection, we can establish the existence result following the proof of Proposition 1.

Note that Propositions 1 and 2 do not prove the existence of an equilibrium cutoff function for auction $A$. Rather, they suggest the following fixed point argument. Proposition 1 shows that, under some conditions, there exist (unique) equilibrium cutoffs for auction $B$. Proposition 2 uses the indifference condition for an equilibrium cutoff function for auction $A$ to characterize it. So given a cutoff function $e$ for auction $A$, we can determine equilibrium cutoffs for auction $B$ as in Proposition 1. Using the indifference condition as in Proposition 2, we can obtain a best-response cutoff function, say $Te$, to $e$. Then any fixed point of the mapping $T$ is an equilibrium cutoff function for auction $A$. In the next proposition, we present a sufficient condition to apply a fixed point theorem and thereby establish the existence of an equilibrium. Before stating the proposition, we introduce some preliminaries.

Let $\bar{f} = \sup_{(x,y)\in[0,1]^2} f(x, y)$. Since $f$ is continuous and $[0, 1]^2$ is compact, $\bar{f}$ is finite. Let

$$\underline{\pi}^A(v^A) = \int_0^1 \int_0^{r^A} (v^A - r^A) f(x, y) dx dy + \int_0^1 \int_{r^A}^{v^A} (v^A - x) f(x, y) dx dy - c^A$$

for all $v^A \in [r^A, 1]$. Note that $\underline{\pi}^A$ is continuous and increasing on $[r^A, 1]$ with $\underline{\pi}^A(r^A) = -c^A < 0$. In the next proposition, we will assume that $c^A < (1 - r^A) - \int_{r^A}^1 (x - r^A) f_A(x) dx$, which means $\underline{\pi}^A(1) > 0$. So under this assumption, there is a unique number $\overline{e}^A \in (r^A, 1)$ such that $\underline{\pi}^A(\overline{e}^A) = 0$. Let

$$\overline{\pi}^A(v^A) = \int_0^1 \int_0^{\overline{e}^A} (v^A - r^A) f(x, y) dx dy + \int_0^1 \int_{\min\{\overline{e}^A, v^A\}}^{v^A} (v^A - x) f(x, y) dx dy - c^A$$

for all $v^A \in [r^A, 1]$. The function $\overline{\pi}^A$ is continuous and increasing on $[r^A, 1]$ with $\underline{\pi}^A(v^A) < \overline{\pi}^A(v^A) < v^A - r^A - c^A$ for all $v^A \in (r^A, 1]$. Hence, there is a unique number $\underline{e}^A \in (r^A + c^A, \overline{e}^A)$ such that $\overline{\pi}^A(\underline{e}^A) = 0$.

**Proposition 3.** *Suppose that*

$$\int_{r^B+c^B}^{1}(y-r^B)f_B(y)dy < c^A < (1-r^A) - \int_{r^A}^{1}(x-r^A)f_A(x)dx$$

*and that*

$$(1-r^B)^2(1-r^A)\overline{e}^A\overline{f}^2 \leq \int_0^{r^B+c^B}\int_{\overline{e}^A}^{1} f(x,y)dxdy \times \int_0^{r^B+c^B}\int_0^{\underline{e}^A} f(x,y)dxdy. \qquad (3)$$

*Then there exists an equilibrium* $(e, e^{oo}, e^{oi}, e^{io}, e^{ii})$ *of the sequential auction format.*

To prove Proposition 3, we apply the Schauder fixed point theorem to the aforementioned mapping $T$. To this end, we need to choose the space of cutoff functions for auction $A$ as a nonempty compact convex subset of a Banach space. To have a Banach space, we consider the set of continuous functions on $[0,1]$ equipped with the uniform norm. As discussed regarding Proposition 2, when $c^A$ is so small that $c^A \leq \Pi^B(1)$, $e$ can be discontinuous at $v^B$ such that $\Pi^B(v^B) = c^A$. So in order to guarantee that $e$ is continuous, we assume that $c^A$ is not so small in Proposition 3.[13] At the same time, we assume that $c^A$ is not so large that a bidder with sufficiently high $v^A$ enters auction $A$ regardless of his $v^B$ (i.e., $e(0) < 1$). Since $e(0) > r^A + c^A$, a bidder enters and does not enter auction $A$ with positive probability, making the conditional distributions $G(\cdot|in)$ and $G(\cdot|out)$ well-defined. The assumption in (3) is made to guarantee the uniqueness of equilibrium cutoffs $(e^{oi}, e^{io})$ for auction $B$ with asymmetric bidders. Without their uniqueness, the mapping $T$ becomes a correspondence. But there is no guarantee that it is convex-valued, and so the Kakutani–Fan–Glicksberg fixed point theorem cannot be applied. Note that, since $\overline{e}^A$ and $\underline{e}^A$ are independent of $c^B$ and $r^B$, the assumption holds true when $c^B \in (0,1)$ is sufficiently small and $r^B \in [0, 1-c^B)$ is sufficiently large.

# 4 Comparison with Simultaneous Auctions

## 4.1 Comparison of Equilibrium Cutoffs

A natural benchmark scenario to our setup is the one where the two auctions are held simultaneously (or equivalently, no information is revealed between the two auctions). In this scenario, there is no interdependence between the two auctions, and thus we look for a symmetric cutoff equilibrium in which a bidder uses a constant, single cutoff in each auction.

[13] An alternative assumption to guarantee the continuity of $e$ is that $r^A = 0$.

Then the equilibrium cutoff $e^k$ for auction $k$ is characterized by

$$(e^k - r^k)F_k(e^k) = c^k, \tag{4}$$

for $k = A, B$. Since the marginal distribution of $v^k$ has support $[0, 1]$, there exists a unique equilibrium cutoff $e^k \in (r^k + c^k, 1)$ for each $k = A, B$. In the following proposition, we compare equilibrium cutoffs for the sequential and simultaneous auction formats.

**Proposition 4.** *Suppose that $(e, e^{oo}, e^{oi}, e^{io}, e^{ii})$ is an equilibrium of the sequential auction format such that $e$ is nonincreasing and $r^B + c^B \leq e^{io} < e^{oo} < e^{ii} < e^{oi} \leq 1$. Suppose that $(e^A, e^B)$ is the symmetric cutoff equilibrium of the simultaneous auction format. Then $e(1) < e^A < e(0)$ and $e^{oo} < e^B < e^{ii}$.*

In the simultaneous auction format, a bidder cannot update his belief about his opponent's valuation of object $B$, and thus he uses the unconditional marginal distribution $F_B$. Since $F_B$ lies between $G(\cdot|in)$ and $G(\cdot|out)$, we have $e^{oo} < e^B < e^{ii}$. Since $e$ is nonincreasing with $e(0) > e(1)$, a bidder's expected payoff in auction $A$ would increase (resp. decrease) if the other bidder adopts the constant cutoff $e(0)$ (resp. $e(1)$) instead of $e$ for auction $A$ in the sequential auction format. This observation leads to $e(1) < e^A < e(0)$. A bidder with a high valuation of object $B$ benefits from entering auction $A$ in the sequential auction format, whereas there are no such gains in the simultaneous auction format. Thus, a bidder with a high valuation of object $B$ enters auction $A$ more aggressively in the sequential auction format than in the simultaneous one. This in turn induces a bidder with a low valuation of object $B$ to refrain from participating in auction $A$ in the sequential auction format. Also, the simultaneous auction format prevents any inference about the opponent's valuation of object $B$, and thus the equilibrium cutoff for auction $B$ in the simultaneous auction format lies in the middle of the equilibrium cutoffs following different histories in the sequential auction format.

### 4.2 Comparison of Auction Performance

As pointed out in the aforementioned fixed point argument and also as can be seen from the proofs of Propositions 1 and 2, equilibrium cutoffs for the two auctions depend on each other in a complicated manner. Moreover, even under the strong assumptions in Proposition 3, there is no guarantee that an equilibrium is unique. Hence, it is difficult to solve for an equilibrium and compare the performance of the two auction formats analytically. So in order to simplify the calculation of equilibrium cutoffs and make the comparison possible, we consider the following particular scenario. The valuations $(v^A, v^B)$ are distributed according to the Clayton copula, as given by (2). The participation costs for the two auctions are the same (i.e., $c^A = c^B$), and we denote the common cost by $c \in (0, 1)$. The objects have no value to the seller, and thus there are no reserve prices (i.e., $r^A = r^B = 0$).

By the defining property of a copula, the marginal distributions of $v^A$ and $v^B$ are uniform on $[0, 1]$. Hence, the equilibrium cutoffs for the simultaneous auction format, characterized by (4), are given by $e^A = e^B = \sqrt{c}$, regardless of $\theta > 0$. As $\theta \to 0$, $F(x, y)$ approaches to $xy$, which corresponds to the case of stochastically independent objects. As $\theta \to \infty$, $F(x, y)$ approaches to $\min\{x, y\}$, which corresponds to the case of identical objects (i.e., $v^A = v^B$). Thus, the parameter $\theta$ can be interpreted as the degree of affiliation between $v^A$ and $v^B$. Although the two limiting cases violate some of our assumptions on the distribution, they are convenient to analyze and the results of the analysis of them are illustrative of the forces created by the sequential auction format. Moreover, by a continuity argument, we can expect that the results would remain close to those from the limiting cases when $\theta$ is sufficiently close to 0 or sufficiently large. Hence, in the following, we study the two limiting cases.

#### 4.2.1 Identical Objects

Suppose that each bidder's valuations are the same across the two objects (i.e., $v_i^A = v_i^B$ for $i = 1, 2$) and that each bidder's common valuation of the objects is uniformly distributed on $[0, 1]$. For any nonincreasing cutoff function $e$ for auction $A$, there exists a unique number $e^* \in [0, 1]$ such that a bidder with $v^A = v^B > e^*$ enters auction $A$ and one with $v^A = v^B < e^*$ does not. So with identical objects, we can describe a cutoff strategy for auction $A$ by a number $e^*$ instead of a function. If a bidder participates in auction $A$, the other bidder believes that the bidder's valuation of object $B$ is distributed uniformly on $[e^*, 1]$. Similarly, if a bidder does not participate in auction $A$, the other bidder believes that the bidder's valuation of object $B$ is distributed uniformly on $[0, e^*]$. The equilibrium cutoffs $e^{ii}$ and $e^{oo}$ for auction $B$ given $e^* \in (0, 1)$ can be obtained as

$$e^{ii} = \frac{e^* + \sqrt{(e^*)^2 + 4(1 - e^*)c}}{2}$$

and

$$e^{oo} = \max\{\sqrt{e^* c}, c\}. \tag{5}$$

When one bidder participates in auction $A$ and the other does not, the participating bidder is believed to have a higher valuation of object $B$ than the non-participating bidder, and thus, at equilibrium, the participating bidder is expected to participate in auction $B$ while the non-participating bidder is not. So the equilibrium cutoffs $e^{io}$ and $e^{oi}$ for auction $B$

given $e^* \in (0,1)$ can be obtained as $e^{io} = c$ and $e^{oi} = \min\{e^* + \sqrt{2(1-e^*)c}, 1\}$. Using the equilibrium behavior in auction $B$ given $e^*$, we can show that $c < e^* < 1$ in equilibrium, and the indifference condition that determines the equilibrium cutoff $e^*$ is given by

$$(e^*)^2 - c + (e^*)^2 - e^*c = e^{oo}e^* + \frac{1}{2}(e^* - e^{oo})^2 - e^*c.$$

Together with $e^{oo} = \sqrt{e^*c}$ in (5), this condition yields the equilibrium cutoff for auction $A$ as

$$e^* = \frac{c + \sqrt{c^2 + 24c}}{6}.$$

It can be verified that $c = e^{io} < e^{oo} < e^* < \sqrt{c} < e^{ii} < e^{oi} \leq 1$ at equilibrium.

We evaluate an auction by the expected revenue of the seller, the expected price of the object conditional on that it is sold, the expected surplus of the bidders, and the expected social surplus. For auction $k = A, B$, we denote these four measures by $R^k$, $P^k$, $BS^k$, and $SS^k$, respectively, while we use $SEQ$ for the sequential auction format and $SIM$ for the simultaneous auction format in the subscripts. For auction $A$ in the sequential auction format, we can obtain the following expressions for these measures:

$$R^A_{SEQ} = (1-e^*)^2 \frac{2e^*+1}{3} = \frac{1}{3} - (e^*)^2 + \frac{2}{3}(e^*)^3, \tag{6}$$

$$P^A_{SEQ} = \frac{(1-e^*)^2}{1-(e^*)^2} \frac{2e^*+1}{3} = \frac{1}{3} - \frac{2(e^*)^2}{3(1+e^*)}, \tag{7}$$

$$BS^A_{SEQ} = \frac{1}{3} + (e^*)^2 - \frac{4}{3}(e^*)^3 - 2(1-e^*)c, \tag{8}$$

$$SS^A_{SEQ} = \frac{2}{3}(1-(e^*)^3) - 2(1-e^*)c. \tag{9}$$

We interpret the participation cost as the value of resources spent by a bidder, not as a fee paid to the seller. Thus, the incurred participation cost is incorporated in the calculation of bidders' surplus and social surplus, but not in the seller's revenue. The measures for

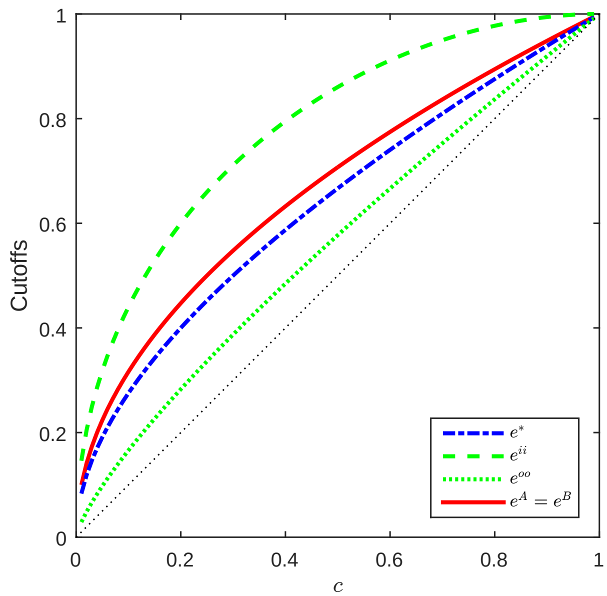



Figure 2: Equilibrium Cutoffs in the Case of Identical Objects

auction $B$ can be computed as follows:

$$
\begin{aligned}
R^B_{SEQ} &= (1-e^{ii})^2\frac{2e^{ii}+1}{3} + (e^* - e^{oo})^2\frac{2e^{oo}+e^*}{3}, \\
P^B_{SEQ} &= \frac{R^B_{SEQ}}{1-(e^{oo})^2-(e^{ii}-e^*)^2}, \\
SS^B_{SEQ} &= \frac{2}{3}(1-(e^{ii})^3) + e^*((e^{ii})^2-(e^*)^2) + \frac{2}{3}((e^*)^3-(e^{oo})^3) \\
&\quad - 2(1-e^{ii})c - 2e^*(e^{ii}-e^{oo})c, \\
BS^B_{SEQ} &= SS^B_{SEQ} - R^B_{SEQ}.
\end{aligned}
$$

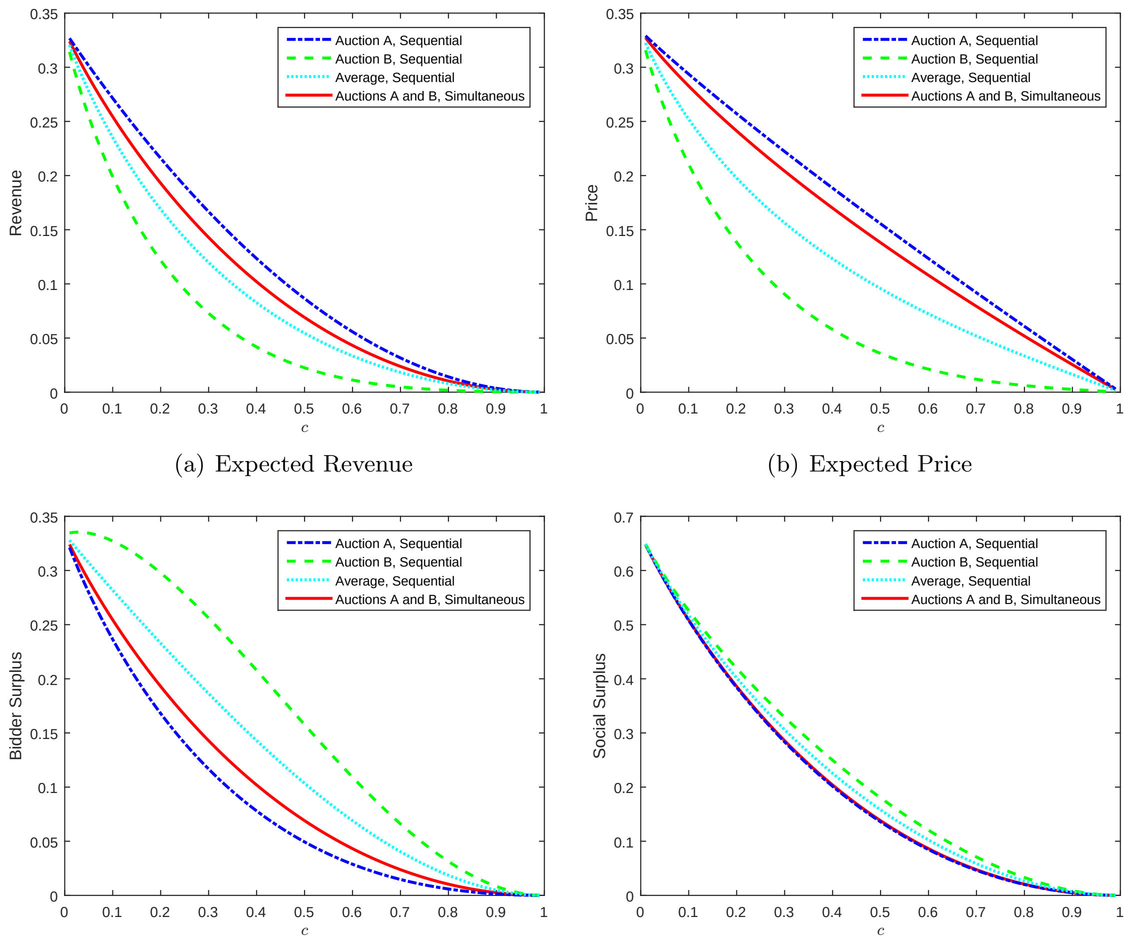
As mentioned before, the equilibrium cutoffs are given by $e^A = e^B = \sqrt{c}$ in the simultaneous auction format. The measures for each auction can be computed as in (6)–(9) with the cutoff $e^*$ replaced by $\sqrt{c}$, which gives

$$R^A_{SIM} = R^B_{SIM} = \frac{1}{3} - c + \frac{2}{3}c\sqrt{c}, \tag{10}$$

$$P^A_{SIM} = P^B_{SIM} = \frac{1}{3} - \frac{2}{3}\frac{c}{1+\sqrt{c}}, \tag{11}$$

$$BS^A_{SIM} = BS^B_{SIM} = \frac{1}{3} - c + \frac{2}{3}c\sqrt{c}, \tag{12}$$

$$SS^A_{SIM} = SS^B_{SIM} = \frac{2}{3} - 2c + \frac{4}{3}c\sqrt{c}. \tag{13}$$



(a) Expected Revenue (b) Expected Price

(c) Expected Bidder Surplus (d) Expected Social Surplus

Figure 3: Four Measures in the Case of Identical Objects

Figure 2 depicts the equilibrium cutoffs $e^*$, $e^{ii}$, $e^{oo}$, and $e^A = e^B = \sqrt{c}$ as $c$ is varied from 0.01 to 0.99. We can verify the relationships $c < e^{oo} < e^* < \sqrt{c} < e^{ii} < 1$. Figure 3 plots the four measures for auctions $A$ and $B$ in the two auction formats. As the participation cost $c$ increases, the equilibrium cutoffs increase as well, making bidders less likely to participate in the auctions. Thus, the expected revenues and the expected prices in both auctions reduce as $c$ increases. Less participation has a beneficial effect on the expected bidder surpluses and the expected social surpluses because it reduces the likelihood of incurring participation cost.[14] However, this beneficial effect is dominated by the loss due to less trade. So the expected bidder surpluses and the expected social surpluses decrease as well, as $c$ increases.

The relationship $e^* < \sqrt{c}$ shows that bidders are more likely to participate in auction $A$ in the sequential auction format than in the simultaneous auction format. This is due to the preemptive entry motive existent in auction $A$ of the sequential auction format, and it leads to the higher expected revenues and the higher expected prices in auction $A$ in the sequential auction format than in the simultaneous one. At the same time, the expected bidder surpluses and the expected social surpluses in auction $A$ are lower in the sequential auction format than in the simultaneous one, because aggressive participation induces a higher total incurred participation cost. The sequential auction format allows more coordination among bidders in auction $B$ based on information disclosed after auction $A$. This weakens competition and results in the lower expected revenues and the lower expected prices in auction $B$ in the sequential auction format than in the simultaneous one. Meanwhile, coordination allows bidders to avoid incurring participation cost, and thus the expected bidder surpluses and the expected social surpluses in auction $B$ are higher in the sequential auction format than in the simultaneous one. Figure 3 also reveals that the impacts of using the sequential auction format on auction $B$ dominate those on auction $A$. Overall, information provided in the sequential auction format improves the expected bidder surpluses and the expected social surpluses, while it reduces the expected revenues and the expected prices. Thus, the seller does not gain by holding auctions sequentially at least in this simple model of identical objects, but if she can extract some of bidders' surpluses, for example, by setting a positive reserve price or an entry fee, she may benefit from using the sequential auction format.

#### 4.2.2 Independent Objects

Suppose that each bidder's valuations of objects $A$ and $B$ are independently and uniformly distributed on $[0, 1]$. In this case, no inference can be made about a bidder's valuation of object $B$ from his entry decision in auction $A$. As a result, there is no preemptive entry

[14] It can be shown that the graph of the expected total incurred participation cost as a function of $c$ is reverse U-shaped for each auction in the two auction formats.

motive in auction $A$ (i.e., $\Pi^B(v^B) = 0$ for all $v^B \in [0,1]$), and the equilibrium cutoff function for auction $A$ is constant at $\sqrt{c}$ (i.e., $e(v^B) = \sqrt{c}$ for all $v^B \in [0,1]$). Because the distribution of $v^B$ is not updated after auction $A$, the equilibrium cutoffs for auction $B$ do not depend on the history and are given by $e^{oo} = e^{oi} = e^{io} = e^{ii} = \sqrt{c}$. So there is no difference between the results of the sequential and simultaneous auction formats in the case of stochastically independent objects. This is because information disclosed in the sequential auction format has no impact on bidders' decisions.

The four measures for auctions $A$ and $B$ are given as in (10)–(13) in both the sequential and simultaneous auction formats. While the expected values are the same for the cases of identical objects and independent objects in the simultaneous auction format, the realized outcomes are different. With identical objects, the outcomes (i.e., who participates, who wins, and how much the winner pays) are the same across the two auctions. In contrast, with independent objects, the outcomes can differ across the two auctions. By comparing the cases of identical objects and independent objects in the sequential auction format, we can expect that the effects of using the sequential auction format on the equilibrium cutoffs and the four measures will be diminished as affiliation between $v^A$ and $v^B$ gets weaker.

## 5 Conclusion and Possible Extensions

In this paper, we studied sequential second-price auctions of two objects with two bidders. The two key elements of our model are affiliation between a bidder's valuations of the two objects and the presence of participation cost. These elements make it possible for a bidder to signal his strength and limit his opponent's entry in the second auction by entering the first auction. As a result, a bidder who is eager to acquire the second object participates in the first auction aggressively. Intensified competition in the first auction due to the signaling motive leads to a higher expected revenue and a higher expected price compared to the simultaneous auction format. On the other hand, competition in the second auction is weakened because bidders can achieve coordination using their entry decisions in the first auction. This results in a higher expected bidder surplus and a higher expected social surplus in the second auction. The signaling incentive also provides an explanation for the declining price anomaly.

We close this paper with possible extensions and modifications of our model. To simplify our analysis on the bidding stage of each auction, we focused on second-price auctions assuming that only entry decisions are revealed after the first auction. However, in the real world, other forms of auctions such as first-price auctions and English auctions are widely used for sequential auctions with more information—for example, the winner and his payment, and all bids—disclosed after each auction. Suppose that first-price auctions are used instead of second-price auctions in our model. Then when bidders are asymmetric in the second auction, a weak bidder tends to bid more aggressively than a strong bidder (see, for example, Krishna, 2009, Proposition 4.4). While a weak bidder participates less aggressively in the second auction, he bids more aggressively when he participates. Thus, the gain from becoming a strong bidder in the second auction will be reduced, leading to less aggressive participation and bidding in the first auction by a bidder with a high valuation of the second object. Next, suppose that, in addition to entry decisions, all or some of bids in the first auction are disclosed before the second auction. Then a bidder can signal his strength not only by entry but also by a high bid. So it will induce bidders to bid more aggressively in the first auction. We can also relax the assumption of stochastic independence of the two bidders' valuations. Suppose instead that the two bidders' valuations of the second object are affiliated. Then a bidder with a high valuation of the second object expects that the other has a high valuation too, and so he will be more cautious about participating in the second auction. This will lessen the gain from signaling. We can consider an alternative scenario where a bidder can observe his opponent's entry decision only when he participates in the auction. In this scenario, a bidder can deliver a signal only when the other bidder participates, and thus the signaling incentive will be reduced again.

Lastly, we can investigate the seller's strategic behavior. The seller can set the reserve prices above her values to increase the revenues. It is possible that the seller chooses the reserve price for the second object depending on her observation from the first auction, but for simplicity, suppose that the reserve price for the second object is independent of the history in the first auction. As the reserve price for the second object gets lower, equilibrium cutoffs for the second auction following different histories will become more dispersed, and this will amplify the gain from signaling. On the other hand, as the reserve price for the first object gets higher, a bidder's entry into the first auction will act as a stronger signal for his strength, and thus a bidder with a high valuation of the second object will be more inclined to enter the first auction. This observation suggests that the seller may benefit from setting a high reserve price and collecting an entry fee in the first auction. In this paper, we fixed the order of the objects put up for sale at the two auctions, but the seller may select the order in her favor. In the simple scenario we analyzed, the simultaneous auction format is superior to the sequential auction format in terms of the total revenue. So studying the question of which order of sale to choose will be more meaningful in an alternative scenario where, for example, the seller sets a reserve price and an entry fee or has a different objective than maximizing the total revenue. Answers to this question will depend on the particular scenario to be analyzed.

# A Appendix: Proofs of Propositions

**Proof of Proposition 1:** Suppose that the function $e : [0,1] \to [0,1]$ is nonincreasing on $[0,1]$, is constant on $[0, r^B + c^B]$ with $e(0) > 0$, and satisfies $e(v^B) < 1$ on some non-degenerate interval. For all $z \in [0,1]$, let

$$G(z|out) = \frac{\int_0^z \int_0^{e(y)} f(x,y)dxdy}{\int_0^1 \int_0^{e(y)} f(x,y)dxdy} \tag{14}$$

and

$$G(z|in) = \frac{\int_0^z \int_{e(y)}^1 f(x,y)dxdy}{\int_0^1 \int_{e(y)}^1 f(x,y)dxdy}. \tag{15}$$

Since $e$ is monotonic, it is continuous almost everywhere. Since $f$ is continuous, the integrals $\int_0^{e(y)} f(x,y)dx$ and $\int_{e(y)}^1 f(x,y)dx$ are well-defined for all $y \in [0,1]$ and continuous in $y$ almost everywhere. Hence, the integrals $\int_0^z \int_0^{e(y)} f(x,y)dxdy$ and $\int_0^z \int_{e(y)}^1 f(x,y)dxdy$ are well-defined for all $z \in [0,1]$ and absolutely continuous in $z$. Since $e(v^B) > 0$ on $[0, r^B + c^B]$, we have $\int_0^1 \int_0^{e(y)} f(x,y)dxdy > 0$. Since $e(v^B) < 1$ on some non-degenerate interval, we have $\int_0^1 \int_{e(y)}^1 f(x,y)dxdy > 0$. Thus, $G(z|out)$ and $G(z|in)$ are well-defined for all $z \in [0,1]$. Note that $G(\cdot|out)$ and $G(\cdot|in)$ are continuous and nondecreasing and satisfy $G(0|out) = G(0|in) = 0$ and $G(1|out) = G(1|in) = 1$. Hence, $G(\cdot|out)$ and $G(\cdot|in)$ can be regarded as cumulative distribution functions. Let $\underline{v}^B = \inf\{v^B \in [0,1] : e(v^B) < 1\}$ and $\overline{v}^B = \sup\{v^B \in [0,1] : e(v^B) > 0\}$. Then $0 \leq \underline{v}^B \leq \overline{v}^B \leq 1$ and $\overline{v}^B \geq r^B + c^B$. The distribution $G(\cdot|out)$ has support $[0, \overline{v}^B]$, while $G(\cdot|in)$ has support $[\underline{v}^B, 1]$.

We show that $G(z|in) < G(z|out)$ for all $z \in (0,1)$. If $z \leq \underline{v}^B$ or $z \geq \overline{v}^B$, the result follows easily. So we can focus on $z \in (\underline{v}^B, \overline{v}^B)$. Since both $G(z|in)$ and $G(z|out)$ are positive for any $z \in (\underline{v}^B, \overline{v}^B)$, $G(z|in) < G(z|out)$ is equivalent to

$$\frac{1}{G(z|in)} - 1 > \frac{1}{G(z|out)} - 1,$$

which in turn is equivalent to

$$\frac{\int_z^1 \int_{e(y)}^1 f(x,y)dxdy}{\int_0^z \int_{e(y)}^1 f(x,y)dxdy} > \frac{\int_z^1 \int_0^{e(y)} f(x,y)dxdy}{\int_0^z \int_0^{e(y)} f(x,y)dxdy}.$$

This can be rewritten as

$$\begin{aligned}
&\int_0^z \int_z^1 \int_0^{e(y)} \int_{e(y')}^1 f(x,y)f(x',y')dx'dxdy'dy \\
&> \int_0^z \int_z^1 \int_0^{e(y')} \int_{e(y)}^1 f(x',y)f(x,y')dx'dxdy'dy. \qquad (16)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^z \int_z^1 \int_0^{e(y)} \int_{e(y')}^1 f(x,y)f(x',y')dx'dxdy'dy \\
&\geq \int_0^z \int_z^1 \int_0^{e(z)} \int_{e(z)}^1 f(x,y)f(x',y')dx'dxdy'dy \\
&> \int_0^z \int_z^1 \int_0^{e(z)} \int_{e(z)}^1 f(x',y)f(x,y')dx'dxdy'dy \\
&\geq \int_0^z \int_z^1 \int_0^{e(y')} \int_{e(y)}^1 f(x',y)f(x,y')dx'dxdy'dy. \qquad (17)
\end{aligned}$$

The first and the third inequalities in (17) hold since $e$ is nonincreasing. Since $e(z) > 0$ for $z$ sufficiently close to 0 and $e(z) < 1$ for $z$ sufficiently close to 1, the second inequality in (17) follows from the affiliation inequality (1). Hence, the inequality in (16) obtains.

Next, we find equilibrium cutoffs for auction $B$ given that both bidders adopt $e$ as the cutoff function for auction $A$. Suppose that neither participates in auction $A$. Then, after auction $A$, each bidder believes that the other bidder's valuation of object $B$ is distributed according to $G(\cdot|out)$. Then a symmetric equilibrium cutoff $e^{oo}$ for auction $B$ after this history is characterized by

$$(e^{oo} - r^B)G(e^{oo}|out) = c^B. \qquad (18)$$

Let $\lambda(v^B) = (v^B - r^B)G(v^B|out)$ for all $v^B \in [r^B, 1]$. Then $\lambda$ is continuous and increasing on $[r^B, 1]$ with $\lambda(r^B + c^B) \leq c^B \leq \lambda(\overline{v}^B)$, and thus there exists a unique symmetric equilibrium cutoff $e^{oo} \in [r^B + c^B, \overline{v}^B]$. If $\overline{v}^B = r^B + c^B$, then we have $e^{oo} = r^B + c^B$. If $\overline{v}^B > r^B + c^B$, then we have $\lambda(r^B + c^B) < c^B < \lambda(\overline{v}^B)$ and thus $e^{oo} \in (r^B + c^B, \overline{v}^B)$.

Suppose that both bidders participate in auction $A$. Then, after auction $A$, each bidder believes that the other bidder's valuation of object $B$ is distributed according to $G(\cdot|in)$. Then a symmetric equilibrium cutoff $e^{ii}$ for auction $B$ after this history is characterized by

$$(e^{ii} - r^B)G(e^{ii}|in) = c^B. \qquad (19)$$

Let $\mu(v^B) = (v^B - r^B)G(v^B|in)$ for all $v^B \in [\max\{r^B, \underline{v}^B\}, 1]$. Then $\mu$ is continuous and increasing on $[\max\{r^B, \underline{v}^B\}, 1]$ with $\mu(\max\{r^B+c^B, \underline{v}^B\}) < c^B < \mu(1)$, and thus there exists a unique symmetric equilibrium cutoff $e^{ii} \in (\max\{r^B+c^B, \underline{v}^B\}, 1)$. Since $G(z|in) < G(z|out)$ for all $z \in (0,1)$, we have $e^{oo} < e^{ii}$.

Now suppose that only one bidder participates in auction $A$. Then, after auction $A$, the nonparticipating bidder believes that the participating bidder's valuation of object $B$ is distributed according to $G(\cdot|in)$, while the participating bidder believes that the nonparticipating bidder's valuation of object $B$ is distributed according to $G(\cdot|out)$. In this case, equilibrium cutoffs $e^{io}$ and $e^{oi}$ such that $e^{io} < e^{oi}$ are characterized by

$$(e^{io} - r^B)G(e^{oi}|out) = c^B, \tag{20}$$

$$(e^{io} - r^B)G(e^{io}|in) + \int_{e^{io}}^{e^{oi}} G(y|in)dy \leq c^B \quad \text{(with equality if } e^{oi} < 1). \tag{21}$$

For all $v \in [e^{oo}, 1]$, let

$$\phi(v) = r^B + \frac{c^B}{G(v|out)}. \tag{22}$$

Since $G(e^{oo}|out) > 0$ and $G(\cdot|out)$ is nondecreasing, $\phi(v)$ is well-defined for all $v \in [e^{oo}, 1]$. The function $\phi$ is continuous, decreasing on $[e^{oo}, \overline{v}^B]$ and constant on $[\overline{v}^B, 1]$ with $\phi(e^{oo}) = e^{oo}$ and $\phi(1) = r^B + c^B$. For all $v \in [e^{oo}, 1]$, let

$$\kappa(v) = (\phi(v) - r^B)G(\phi(v)|in) + \int_{\phi(v)}^{v} G(y|in)dy. \tag{23}$$

Since $\phi$ and $G(\cdot|in)$ are continuous, $\kappa$ is continuous. Since $\phi(e^{oo}) = e^{oo}$, we have $\kappa(e^{oo}) = (e^{oo} - r^B)G(e^{oo}|in) < c^B$. If $\kappa(1) < c^B$, then $e^{oi} = 1$ and $e^{io} = r^B + c^B$ are equilibrium cutoffs. If $\kappa(1) \geq c^B$, then by the intermediate value theorem there exists $v^* \in (e^{oo}, 1]$ such that $\kappa(v^*) = c^B$, and $e^{oi} = v^*$ and $e^{io} = \phi(v^*)$ are equilibrium cutoffs. Hence, there exist equilibrium cutoffs $e^{io}$ and $e^{oi}$ such that $e^{io} < e^{oi}$. If $\overline{v}^B = r^B + c^B$, then we have $e^{io} = e^{oo} = r^B+c^B$ and $e^{oi} = 1$. If $\overline{v}^B > r^B+c^B$, then we have $r^B+c^B \leq e^{io} < e^{oo} < e^{oi} \leq 1$.

To show that $e^{ii} < e^{oi}$, suppose to the contrary that $e^{ii} \geq e^{oi}$. Then $e^{oi} < 1$, and thus $G(e^{oi}|in) > 0$. Since $G(\cdot|in)$ is nondecreasing, we have

$$\begin{aligned}
c^B &= (e^{io} - r^B)G(e^{io}|in) + \int_{e^{io}}^{e^{oi}} G(y|in)dy \\
&\leq (e^{io} - r^B)G(e^{io}|in) + (e^{oi} - e^{io})G(e^{oi}|in) < (e^{oi} - r^B)G(e^{oi}|in) \\
&\leq (e^{ii} - r^B)G(e^{ii}|in) = c^B,
\end{aligned}$$

a contradiction. Hence, any equilibrium cutoffs $(e^{oo}, e^{oi}, e^{io}, e^{ii})$ for auction $B$ such that

$e^{io} < e^{oi}$ satisfy $r^B + c^B \le e^{io} \le e^{oo} < e^{ii} < e^{oi} \le 1$ with $e^{io} < e^{oo}$ if and only if $\overline{v}^B > r^B + c^B$.

To prove the uniqueness of equilibrium cutoffs for auction $B$, suppose that $e(0) < 1$, $f$ is nonincreasing in the second argument, and $\int_{e(y)}^1 f(x,y)dx$ is nonincreasing in $y$. Note that $G(\cdot|out)$ and $G(\cdot|in)$ are differentiable almost everywhere with their derivatives given by

$$G'(z|out) = \frac{\int_0^{e(z)} f(x,z)dx}{\int_0^1 \int_0^{e(y)} f(x,y)dxdy}$$

and

$$G'(z|in) = \frac{\int_{e(z)}^1 f(x,z)dx}{\int_0^1 \int_{e(y)}^1 f(x,y)dxdy}$$

almost everywhere. Since $f$ is nonincreasing in the second argument, $G'(\cdot|out)$ is nonincreasing, and thus $G(\cdot|out)$ is concave on $[0,1]$. Then by Proposition 5(i) of Tan and Yilankaya (2006), there are no equilibrium cutoffs $e^{io}$ and $e^{oi}$ such that $e^{io} \ge e^{oi}$.

The functions $\phi$ and $\kappa$ are differentiable almost everywhere, and the derivative of $\kappa$ is is given by

$$\begin{aligned}\kappa'(v) &= (\phi(v) - r^B)G'(\phi(v)|in)\phi'(v) + G(v|in)\\ &= G(v|in)\left[1 - (\phi(v) - r^B)^2 \frac{G'(\phi(v)|in)}{G(v|in)} \frac{G'(v|out)}{G(v|out)}\right]\end{aligned} \tag{24}$$

almost everywhere. Since $G(\cdot|out)$ is concave, for all $v \in (e^{oo}, 1)$, we have

$$\frac{G'(v|out)}{G(v|out)} \le \frac{1}{v} < \frac{1}{\phi(v)} \le \frac{1}{\phi(v) - r^B}.$$

Since $e(0) < 1$, we have $\underline{v}^B = 0$. Since $\int_{e(y)}^1 f(x,y)dx$ is nonincreasing in $y$, $G'(\cdot|in)$ is nonincreasing, and thus $G(\cdot|in)$ is concave on $[0,1]$. Then for all $v \in (e^{oo}, 1)$, we have

$$\frac{G'(\phi(v)|in)}{G(v|in)} < \frac{G'(\phi(v)|in)}{G(\phi(v)|in)} \le \frac{1}{\phi(v)} \le \frac{1}{\phi(v) - r^B}.$$

By (24), $\kappa'(v) > 0$ almost everywhere on $(e^{oo}, 1)$, and thus $\kappa$ is increasing on $[e^{oo}, 1]$. Hence, equilibrium cutoffs $e^{io}$ and $e^{oi}$ are uniquely given by $e^{oi} = \sup\{v^B \in [e^{oo}, 1] : \kappa(v^B) < c^B\}$ and $e^{io} = \phi(e^{oi})$. $\square$

**Proof of Proposition 2:** Suppose that $(e, e^{oo}, e^{oi}, e^{io}, e^{ii})$ is an equilibrium such that $e$ is continuous almost everywhere and $r^B + c^B \leq e^{io} \leq e^{oo} < e^{ii} < e^{oi} \leq 1$. Since $e$ is continuous almost everywhere, we can define $G(\cdot|out)$ and $G(\cdot|in)$ as in (14) and (15), respectively, as long as $\int_0^1 \int_0^{e(y)} f(x,y)dxdy > 0$ and $\int_0^1 \int_{e(y)}^1 f(x,y)dxdy > 0$. Given $G(\cdot|out)$ and $G(\cdot|in)$, the cutoffs $(e^{oo}, e^{oi}, e^{io}, e^{ii})$ are characterized by (18)–(21).

Let $\tilde{e}(v^B) = \max\{e(v^B), r^A\}$ for all $v^B \in [0,1]$. Consider a bidder with valuations $(v^A, v^B)$. Suppose that the other bidder uses $e$ as the cutoff function for auction $A$ and that the two bidders make participation decisions for auction $B$ according to the cutoffs $(e^{oo}, e^{oi}, e^{io}, e^{ii})$. Let $\pi^A(v^A)$ be the bidder's expected payoff in auction $A$ provided that he participates. Then we have

$$\pi^A(v^A) = \begin{cases} -c^A & \text{if } 0 \leq v^A \leq r^A, \\ \int_0^1 \int_0^{\tilde{e}(y)} (v^A - r^A) f(x,y)dxdy & \\ + \int_0^1 \int_{\min\{\tilde{e}(y), v^A\}}^{v^A} (v^A - x) f(x,y)dxdy - c^A & \text{if } r^A < v^A \leq 1. \end{cases} \tag{25}$$

Let $\pi^B(v^B|in)$ be the bidder's expected payoff in auction $B$ after participating in auction $A$ (before learning the other bidder's entry decision in auction $A$), and let $\pi^B(v^B|out)$ be that after not participating in auction $A$. Then we have

$$\pi^B(v^B|in) = \begin{cases} 0 & \text{if } 0 \leq v^B \leq e^{io}, \\ \int_0^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f(x,y)dxdy - c^B \int_0^1 \int_0^{e(y)} f(x,y)dxdy & \\ & \text{if } e^{io} < v^B \leq e^{ii}, \\ \int_0^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f(x,y)dxdy + \int_{e^{oi}}^{\max\{e^{oi}, v^B\}} \int_0^{e(y)} (v^B - y) f(x,y)dxdy & \\ + \int_0^{e^{ii}} \int_{e(y)}^1 (v^B - r^B) f(x,y)dxdy + \int_{e^{ii}}^{v^B} \int_{e(y)}^1 (v^B - y) f(x,y)dxdy - c^B & \\ & \text{if } e^{ii} < v^B \leq 1, \end{cases}$$

and

$$\pi^B(v^B|out) = \begin{cases} 0 & \text{if } 0 \leq v^B \leq e^{oo}, \\ \int_0^{e^{oo}} \int_0^{e(y)} (v^B - r^B) f(x,y)dxdy + \int_{e^{oo}}^{v^B} \int_0^{e(y)} (v^B - y) f(x,y)dxdy & \\ -c^B \int_0^1 \int_0^{e(y)} f(x,y)dxdy & \text{if } e^{oo} < v^B \leq e^{oi}, \\ \int_0^{e^{oo}} \int_0^{e(y)} (v^B - r^B) f(x,y)dxdy + \int_{e^{oo}}^{v^B} \int_0^{e(y)} (v^B - y) f(x,y)dxdy & \\ + \int_0^{e^{io}} \int_{e(y)}^1 (v^B - r^B) f(x,y)dxdy + \int_{e^{io}}^{v^B} \int_{e(y)}^1 (v^B - y) f(x,y)dxdy - c^B & \\ & \text{if } e^{oi} < v^B \leq 1. \end{cases}$$

Let $\Pi^B(v^B) = \pi^B(v^B|in) - \pi^B(v^B|out)$ for all $v^B \in [0,1]$. Then we have

$$\Pi^B(v^B) = \begin{cases} 0 & \text{if } 0 \le v^B \le e^{io}, \\ \int_0^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f(x,y) dx dy - c^B \int_0^1 \int_0^{e(y)} f(x,y) dx dy & \\ & \text{if } e^{io} < v^B \le e^{oo}, \\ \int_{v^B}^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f(x,y) dx dy + \int_{e^{oo}}^{v^B} \int_0^{e(y)} (y - r^B) f(x,y) dx dy & \\ & \text{if } e^{oo} < v^B \le e^{ii}, \\ \int_{v^B}^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f(x,y) dx dy + \int_{e^{oo}}^{v^B} \int_0^{e(y)} (y - r^B) f(x,y) dx dy & \\ + \int_0^{e^{ii}} \int_{e(y)}^1 (v^B - r^B) f(x,y) dx dy + \int_{e^{ii}}^{v^B} \int_{e(y)}^1 (v^B - y) f(x,y) dx dy & \\ - c^B \int_0^1 \int_{e(y)}^1 f(x,y) dx dy & \text{if } e^{ii} < v^B \le e^{oi}, \\ \int_{e^{oo}}^{e^{oi}} \int_0^{e(y)} (y - r^B) f(x,y) dx dy + \int_{e^{io}}^{e^{ii}} \int_{e(y)}^1 (y - r^B) f(x,y) dx dy & \\ & \text{if } e^{oi} < v^B \le 1. \end{cases} \tag{26}$$

Since $e$ is continuous almost everywhere, the integrals in (25) and (26) are well-defined. Since $\lim_{v^A \to (r^A)^+} \pi^A(v^A) = -c^A$, $\pi^A$ is continuous. It is piecewise differentiable with

$$\frac{d\pi^A(v^A)}{dv^A} = \begin{cases} 0 & \text{if } 0 \le v^A < r^A, \\ \int_0^1 \int_0^{\max\{\tilde{e}(y), v^A\}} f(x,y) dx dy & \text{if } r^A < v^A \le 1. \end{cases} \tag{27}$$

Thus, $\pi^A$ is constant on $[0, r^A]$ and increasing on $[r^A, 1]$. Using (18)–(21), we can show that $\Pi^B$ is continuous. It is piecewise differentiable with

$$\frac{d\Pi^B(v^B)}{dv^B} = \begin{cases} 0 & \text{if } 0 \le v^B < e^{io}, \\ \int_0^{e^{oi}} \int_0^{e(y)} f(x,y) dx dy & \text{if } e^{io} < v^B < e^{oo}, \\ \int_{v^B}^{e^{oi}} \int_0^{e(y)} f(x,y) dx dy & \text{if } e^{oo} < v^B < e^{ii}, \\ \int_{v^B}^{e^{oi}} \int_0^{e(y)} f(x,y) dx dy + \int_0^{v^B} \int_{e(y)}^1 f(x,y) dx dy & \text{if } e^{ii} < v^B < e^{oi}, \\ 0 & \text{if } e^{oi} < v^B \le 1. \end{cases} \tag{28}$$

Thus, $\Pi^B$ is constant on $[0, e^{io}]$ and $[e^{oi}, 1]$ and nondecreasing on $[e^{io}, e^{oi}]$.

The bidder prefers entering auction $A$ if and only if $\pi^A(v^A) + \Pi^B(v^B) > 0$. Since $e$ is an equilibrium cutoff function for auction $A$, it satisfies

$$\pi^A(e(v^B)) + \Pi^B(v^B) \begin{cases} \le 0 & \text{if } e(v^B) > 0, \\ \ge 0 & \text{if } e(v^B) < 1, \end{cases} \tag{29}$$

for all $v^B \in [0,1]$. Since $\pi^A$ is continuous and increasing on $[r^A, 1]$ and $\Pi^B$ is continuous and nondecreasing on $[0,1]$, the function $e$ is continuous and nonincreasing on $\{v^B \in [0,1] : e(v^B) > r^A\}$. Since $\Pi^B$ is constant on $[0, e^{io}]$ with $\Pi^B(0) = 0$, $e$ is also constant on

$[0, e^{io}]$, regardless of whether $e(0) = 1$ or $e(0) < 1$. We show that $e(0) > r^A + c^A$. If $e(0) = 1$, the result holds since $r^A < 1 - c^A$. So we consider the case of $e(0) < 1$. Since $\tilde{e}$ is nonincreasing, we have $\tilde{e}(v^B) \leq e(0) < 1$ for all $v^B \in [0, 1]$ and thus $d\pi^A(v^A)/dv^A = \int_0^1 \int_0^{\max\{\tilde{e}(y), v^A\}} f(x, y)dxdy < 1$ for all $v^A \in (r^A, 1)$. Hence, $\pi^A(v^A) < v^A - r^A - c^A$ for all $v^A \in (r^A, 1)$. Since $\pi^A(e(0)) = 0$, it follows that $e(0) > r^A + c^A$.

By comparing $\Pi^B(1)$ and $c^A$, we can consider the following three cases.

*Case 1.* $\Pi^B(1) < c^A$

In this case, (29) implies that $e(v^B) > r^A$ for all $v^B \in [0, 1]$. Then, from (28), we can see that $\Pi^B$ is increasing on $[e^{io}, e^{oi}]$ and constant on $[e^{oi}, 1]$. Thus, $e$ is decreasing on $[e^{io}, e^{oi}] \cap \{v^B \in [0, 1] : e(v^B) < 1\}$ and constant on $[e^{oi}, 1]$. Since $e$ is continuous on $[0, 1]$, $\pi^A$ is continuously differentiable with positive derivative at any $v^A \in (r^A, 1)$, while $\Pi^B$ is continuously differentiable except at $e^{io}$, $e^{oo}$, $e^{ii}$, and $e^{oi}$. Hence, by the implicit function theorem, $e$ is continuously differentiable on $\{v^B \in [0, 1] : e(v^B) < 1\}$ except at $e^{io}$, $e^{oo}$, $e^{ii}$, and $e^{oi}$.

*Case 2.* $\Pi^B(1) > c^A$

Since $\Pi^B$ is continuous and nondecreasing with $\Pi^B(e^{io}) = 0$, the set $\{v^B \in [0, 1] : \Pi^B(v^B) = c^A\}$ is an interval $[v_1, v_2]$ where $e^{io} < v_1 \leq v_2 < e^{oi}$. Then (29) implies that $e(v^B) > r^A$ for all $v^B \in [0, v_1)$, $\lim_{v^B \to v_1^-} e(v^B) = r^A$, $0 \leq e(v^B) \leq r^A$ for all $v^B \in [v_1, v_2]$, and $e(v^B) = 0$ for all $v^B \in (v_2, 1]$. Since $e$ is continuous on $\{v^B \in [0, 1] : e(v^B) > r^A\}$ and $e(0) > r^A + c^A > 0$, we can see from (28) that $\Pi^B$ is increasing on $[e^{io}, e^{oo}]$ and $[e^{ii}, e^{oi}]$ and that it is increasing around $v^B \in (e^{oo}, e^{ii})$ if $e(v^B) > r^A$. Hence, $e$ is decreasing on $[e^{io}, v_1) \cap \{v^B \in [0, 1] : e(v^B) < 1\}$. Since $e$ is continuous on $[0, v_1)$, $\Pi^B$ is continuously differentiable on $(0, v_1)$ except at $e^{io}$, $e^{oo}$, $e^{ii}$, and $e^{oi}$. Hence, $e$ is continuously differentiable on $\{v^B \in [0, 1] : r^A < e(v^B) < 1\}$ except at $e^{io}$, $e^{oo}$, $e^{ii}$, and $e^{oi}$.

First, we consider the case where $[v_1, v_2]$ is non-degenerate (i.e., $v_1 < v_2$). Note that $d\Pi^B(v^B)/dv^B$ is nonincreasing on $[e^{oo}, e^{ii}]$, and thus we have $e^{oo} \leq v_1 < v_2 = e^{ii}$ and $e(v^B) = 0$ for almost every $v^B \in [v_1, v_2]$. Then $G(\cdot|out)$ has support $[0, v_1]$, and so we write $\overline{v}^B = v_1$. Suppose that $e^{oo} = \overline{v}^B$. Then from (18), we obtain $e^{oo} = r^B + c^B$. Since $r^B + c^B \leq e^{io} \leq e^{oo}$, we have $e^{io} = e^{oo}$. However, this contradicts $\Pi^B(e^{io}) = 0$ and $\Pi^B(v_1) = c^A$. Hence, we have $e^{oo} < \overline{v}^B$.

Next, we consider the case where $[v_1, v_2]$ is degenerate (i.e., $v_1 = v_2$). Again, $G(\cdot|out)$ has support $[0, v_1]$, and so we write $\overline{v}^B = v_1$. Suppose that $\overline{v}^B < e^{ii}$. Since $e(v^B) = 0$ for all $v^B \in (\overline{v}^B, 1]$, $\Pi^B$ is constant on $[\overline{v}^B, e^{ii}]$. Then $\Pi^B(v^B) = c^A$ for all $v^B \in [\overline{v}^B, e^{ii}]$, which is a contradiction. So we have $\overline{v}^B \geq e^{ii}$.

To sum up, when $[v_1, v_2]$ is non-degenerate, we have $[v_1, v_2] = [\overline{v}^B, e^{ii}]$ where $\overline{v}^B \in (e^{oo}, e^{ii})$, and $e(v^B) = 0$ for almost every $v^B \in [\overline{v}^B, e^{ii}]$. When $[v_1, v_2]$ is degenerate, we have $[v_1, v_2] = \{\overline{v}^B\}$ where $\overline{v}^B \in [e^{ii}, e^{oi})$. In either case, we have $G(e^{oi}|out) = 1$, and thus we obtain $e^{io} = r^B + c^B$ by (20).

*Case 3.* $\Pi^B(1) = c^A$

Since $\Pi^B$ is constant on $[e^{oi}, 1]$, the set $\{v^B \in [0,1] : \Pi^B(v^B) = c^A\}$ is equal to $[e^{oi}, 1]$. Then (29) implies that $e(v^B) > r^A$ for all $v^B \in [0, e^{oi})$, $\lim_{v^B \to (e^{oi})^-} e(v^B) = r^A$, and $0 \leq e(v^B) \leq r^A$ for all $v^B \in [e^{oi}, 1]$. Following the same argument as in Case 2, we can show that $e$ is decreasing on $[e^{io}, e^{oi}) \cap \{v^B \in [0,1] : e(v^B) < 1\}$ and that it is continuously differentiable on $\{v^B \in [0,1] : r^A < e(v^B) < 1\}$ except at $e^{io}$, $e^{oo}$, $e^{ii}$, and $e^{oi}$.

Suppose that $e^{io} = e^{oo}$. From (18) and (20), we have $G(e^{oo}|out) = G(e^{oi}|out)$, which implies $e(v^B) = 0$ for almost every $v^B \in (e^{oo}, e^{oi})$. However, this cannot occur in any of the above three cases. Hence, it follows that $e^{io} < e^{oo}$.

We have $e(v^B) = 1$ if and only if $\pi^A(1) + \Pi^B(v^B) \leq 0$. Suppose that $e(v^B) = 1$ for some $v^B \in [0,1]$. Since $\Pi^B$ is continuous and nondecreasing, the set $\{v^B \in [0,1] : \Pi^B(v^B) \leq -\pi^A(1)\}$ is an interval $[0, v_3]$ where $0 < v_3 \leq 1$. We show that $e(1) < 1$. Suppose to the contrary that $e(1) = 1$. Since $e$ is nonincreasing on $\{v^B \in [0,1] : e(v^B) > r^A\}$, we have $e(v^B) = 1$ for all $v^B \in [0,1]$. Then we have $\pi^A(v^A) = v^A - r^A - c^A$ for all $v^A \in [r^A, 1]$, which yields $\pi^A(e(1)) = 1 - r^A - c^A > 0$. Since $\Pi^B(1) \geq 0$, we have $\pi^A(e(1)) + \Pi^B(1) > 0$, which is a contradiction. Hence, we have $v_3 < 1$. Then $G(\cdot|in)$ has support $[v_3, 1]$, and so we write $\underline{v}^B = v_3$. From (19), we obtain $e^{ii} > \underline{v}^B$. Since $\Pi^B$ is constant on $[0, e^{io}]$, we have $\underline{v}^B \geq e^{io}$. $\square$

**Proof of Proposition 3:** Let

$$L = \frac{1}{\int_0^{r^B + c^B} \int_0^{\underline{e}^A} f(x,y)\,dx\,dy}.$$

Let $C[0,1]$ be the set of continuous functions from $[0,1]$ to $\mathbb{R}$, equipped with the uniform norm $\|\cdot\|$. Let $K \subset C[0,1]$ be the set of functions $e : [0,1] \to [0,1]$ such that $e$ is Lipschitz continuous with Lipschitz constant $L$, is nonincreasing, is constant on $[0, r^B + c^B]$, and satisfies $\underline{e}^A \leq e(0) \leq \overline{e}^A$ and $e(1) \geq r^A$. Note that $K$ is nonempty and convex. Moreover, $K$ is uniformly bounded, closed, and equicontinuous, and thus it is compact by the Arzelà–Ascoli theorem.

Choose any function $e \in K$, and define $G(\cdot|out)$ and $G(\cdot|in)$ by (14) and (15), respectively. Since $e$ is monotonic and $0 < \underline{e}^A \leq e(0) \leq \overline{e}^A < 1$, $G(\cdot|out)$ and $G(\cdot|in)$ are well-defined with supports $[0, \overline{v}^B]$ and $[0,1]$, respectively, and satisfy $G(z|in) < G(z|out)$ for all $z \in (0,1)$, as in the proof of Proposition 1.

From (24), we can see that

$$\begin{aligned}\kappa'(v) &= G(v|in)\left[1-(\phi(v)-r^B)^2\frac{G'(\phi(v)|in)}{G(v|in)}\frac{G'(v|out)}{G(v|out)}\right] \\ &= G(v|in)\left[1-(\phi(v)-r^B)^2\frac{\int_{e(\phi(v))}^1 f(x,\phi(v))dx}{\int_0^v\int_{e(y)}^1 f(x,y)dxdy}\frac{\int_0^{e(v)} f(x,v)dx}{\int_0^v\int_0^{e(y)} f(x,y)dxdy}\right]\end{aligned}$$

for almost every $v \in (e^{oo},1)$. Since $e^{oo} < v < 1$ and $r^B + c^B \leq \phi(v) < e^{oo}$, we have $0 < \phi(v) - r^B < 1 - r^B$, $0 < \int_{e(\phi(v))}^1 f(x,\phi(v))dx \leq (1-r^A)\bar{f}$, $0 < \int_0^{e(v)} f(x,v)dx \leq \bar{e}^A\bar{f}$, $\int_0^v\int_{e(y)}^1 f(x,y)dxdy \geq \int_0^{r^B+c^B}\int_{\bar{e}^A}^1 f(x,y)dxdy > 0$, and $\int_0^v\int_0^{e(y)} f(x,y)dxdy \geq \int_0^{r^B+c^B}\int_0^{\underline{e}^A} f(x,y)dxdy > 0$. Hence, the assumption that

$$(1-r^B)^2(1-r^A)\bar{e}^A\bar{f}^2 \leq \int_0^{r^B+c^B}\int_{\bar{e}^A}^1 f(x,y)dxdy \times \int_0^{r^B+c^B}\int_0^{\underline{e}^A} f(x,y)dxdy,$$

implies that $\kappa$ is increasing on $[e^{oo},1]$. Then, the conditions (18)–(21) determine equilibrium cutoffs $(e^{oo},e^{oi},e^{io},e^{ii})$ for auction $B$ uniquely, and they satisfy $r^B + c^B \leq e^{io} < e^{oo} < e^{ii} < e^{oi} \leq 1$, since $\bar{v}^B > r^B + c^B$.

Given the cutoff function $e$ for auction $A$ and the equilibrium cutoffs $(e^{oo},e^{oi},e^{io},e^{ii})$ for auction $B$, we can define $\pi^A$ and $\Pi^B$ by (25) and (26), respectively. Since $r^B + c^B \leq e^{io} < e^{oo} < e^{ii} < e^{oi} \leq 1$, we have

$$\Pi^B(1) \leq \int_{r^B+c^B}^1\int_0^1 (y-r^B)f(x,y)dxdy = \int_{r^B+c^B}^1 (y-r^B)f_B(y)dy.$$

By the assumption that $\int_{r^B+c^B}^1 (y-r^B)f_B(y)dy < c^A$, we have $\Pi^B(1) < c^A$.

For all $v^B \in [0,1]$, define $Te(v^B)$ by

$$\pi^A(Te(v^B)) + \Pi^B(v^B) = 0. \tag{30}$$

Note that $0 \leq \Pi^B(v^B) < c^A$ for all $v^B \in [0,1]$ and that $\pi^A$ is increasing on $[r^A,1]$ with $\pi^A(r^A) = -c^A$ and $\pi^A(1) > 0$. Hence, for each $v^B \in [0,1]$, there exists a unique number $Te(v^B) \in (r^A,1)$ satisfying the condition (30). So we have $Te(1) > r^A$. Note also that $Te$ is continuous and nonincreasing and that it is constant on $[0,e^{io}]$, which includes $[0,r^B+c^B]$. Since $\Pi^B(0) = 0$, we have $\pi^A(Te(0)) = 0$, and thus $\underline{e}^A \leq Te(0) \leq \bar{e}^A$. Since $e$ is constant on $[0,r^B+c^B]$ with $e(0) \geq \underline{e}^A$, we can see from (27) that

$$\left|\frac{d\pi^A(v^A)}{dv^A}\right| \geq \int_0^{r^B+c^B}\int_0^{\underline{e}^A} f(x,y)dxdy$$

for all $v^A > r^A$. Also, from (28), we can see that

$$\left|\frac{d\Pi^B(v^B)}{dv^B}\right| \leq 1$$

wherever $\Pi^B$ is differentiable. Note that $\pi^A$ is continuously differentiable with positive derivative at any $v^A \in (r^A, 1)$ and that $\Pi^B$ is piecewise continuously differentiable. By the implicit function theorem, $Te$ is piecewise continuously differentiable, and its derivative is given by

$$\frac{dTe(v^B)}{dv^B} = -\frac{\frac{d\Pi^B(v^B)}{dv^B}}{\frac{d\pi^A(Te(v^B))}{dv^A}}$$

wherever it is differentiable. Hence, we have

$$\left|\frac{dTe(v^B)}{dv^B}\right| \leq \frac{1}{\int_0^{r^B+c^B}\int_0^{\underline{e}^A} f(x,y)dxdy} = L$$

for all $v^B \in (0,1) \setminus \{e^{io}, e^{oo}, e^{ii}, e^{oi}\}$, and $Te$ is Lipschitz continuous with Lipschitz constant $L$. So $Te$ belongs to the set $K$.

We can regard $T$ as a function from $K$ to itself. Moreover, $T$ is continuous (see Lemma 1 below). Then, by the Schauder fixed point theorem (see, for example, Corollary 17.56 of Aliprantis and Border, 2006), there exists a fixed point $e^*$ of $T$ such that $Te^* = e^*$.

For any fixed point $e^*$ of $T$, consider the cutoff strategy $(e^*, e^{oo}, e^{oi}, e^{io}, e^{ii})$ where the cutoffs $(e^{oo}, e^{oi}, e^{io}, e^{ii})$ are the equilibrium cutoffs for auction $B$ given the cutoff function $e^*$ for auction $A$. Suppose that $e^*$ is prescribed for auction $A$ in an equilibrium. When a bidder follows $e^*$, he enters auction $A$ with positive probability and does not enter with positive probability. Thus, even when he deviates from $e^*$ in auction $A$, the belief of the other bidder is still determined by $e^*$, and the equilibrium cutoffs for auction $B$ are not affected by his deviation. The condition (30) means that it is optimal for a bidder to follow $e^*$ in auction $A$ given that the other bidder uses $e^*$ for auction $A$ and that the equilibrium cutoffs for auction $B$ are determined by $e^*$. Therefore, $(e^*, e^{oo}, e^{oi}, e^{io}, e^{ii})$ is an equilibrium. $\square$

**Lemma 1.** *The function $T : K \to K$ is continuous.*

*Proof.* Choose any sequence $\{e_n\}_{n=1}^{\infty}$ in $K$ that converges uniformly to some function $e$. Since $K$ is closed, $e$ belongs to $K$. From $e$, we can derive $G(\cdot|in)$, $G(\cdot|out)$, $(e^{oo}, e^{oi}, e^{io}, e^{ii})$, $\pi^A$, $\Pi^B$, and $Te$ uniquely. Similarly, for each $n = 1, 2, \ldots$, we can derive these objects uniquely from $e_n$, and we use the subscript $n$ to signify the derivation from $e_n$. We show that the sequence $\{Te_n\}_{n=1}^{\infty}$ converges uniformly to $Te$ following four steps.

*Step 1.* The sequences $\{G_n(\cdot|in)\}_{n=1}^{\infty}$ and $\{G_n(\cdot|out)\}_{n=1}^{\infty}$ converge uniformly to $G(\cdot|in)$ and $G(\cdot|out)$, respectively.

For notational simplicity, we use $f$ instead of $f(x,y)$ as the integrand in the proof of this lemma. For any $n = 1, 2, \ldots$ and any $z \in [0,1]$, we have

$$
\begin{aligned}
&|G_n(z|in) - G(z|in)| = \left| \frac{\int_0^z \int_{e_n(y)}^1 f\,dxdy}{\int_0^1 \int_{e_n(y)}^1 f\,dxdy} - \frac{\int_0^z \int_{e(y)}^1 f\,dxdy}{\int_0^1 \int_{e(y)}^1 f\,dxdy} \right| \\
&= \left| \frac{\int_0^z \int_{e_n(y)}^1 f\,dxdy \times \int_0^1 \int_{e(y)}^1 f\,dxdy - \int_0^z \int_{e(y)}^1 f\,dxdy \times \int_0^1 \int_{e_n(y)}^1 f\,dxdy}{\int_0^1 \int_{e_n(y)}^1 f\,dxdy \times \int_0^1 \int_{e(y)}^1 f\,dxdy} \right| \\
&= \left| \frac{\int_0^z \int_{e_n(y)}^1 f\,dxdy \times \int_z^1 \int_{e(y)}^1 f\,dxdy - \int_0^z \int_{e(y)}^1 f\,dxdy \times \int_z^1 \int_{e_n(y)}^1 f\,dxdy}{\int_0^1 \int_{e_n(y)}^1 f\,dxdy \times \int_0^1 \int_{e(y)}^1 f\,dxdy} \right| \\
&= \frac{1}{\left|\int_0^1 \int_{e_n(y)}^1 f\,dxdy \times \int_0^1 \int_{e(y)}^1 f\,dxdy\right|} \times \\
&\left| \int_0^z \int_{e_n(y)}^1 f\,dxdy \times \int_z^1 \int_{e(y)}^1 f\,dxdy - \int_0^z \int_{e_n(y)}^1 f\,dxdy \times \int_z^1 \int_{e_n(y)}^1 f\,dxdy \right. \\
&\left. + \int_0^z \int_{e_n(y)}^1 f\,dxdy \times \int_z^1 \int_{e_n(y)}^1 f\,dxdy - \int_0^z \int_{e(y)}^1 f\,dxdy \times \int_z^1 \int_{e_n(y)}^1 f\,dxdy \right| \\
&= \frac{\left|\int_0^z \int_{e_n(y)}^1 f\,dxdy \times \int_z^1 \int_{e(y)}^{e_n(y)} f\,dxdy - \int_0^z \int_{e(y)}^{e_n(y)} f\,dxdy \times \int_z^1 \int_{e_n(y)}^1 f\,dxdy\right|}{\left|\int_0^1 \int_{e_n(y)}^1 f\,dxdy \times \int_0^1 \int_{e(y)}^1 f\,dxdy\right|} \\
&\leq \frac{\left|\int_0^z \int_{e_n(y)}^1 f\,dxdy\right| \left|\int_z^1 \int_{e(y)}^{e_n(y)} f\,dxdy\right| + \left|\int_0^z \int_{e(y)}^{e_n(y)} f\,dxdy\right| \left|\int_z^1 \int_{e_n(y)}^1 f\,dxdy\right|}{\left|\int_0^1 \int_{e_n(y)}^1 f\,dxdy\right| \left|\int_0^1 \int_{e(y)}^1 f\,dxdy\right|} \\
&\leq \frac{\left|\int_z^1 \int_{e(y)}^{e_n(y)} f\,dxdy\right| + \left|\int_0^z \int_{e(y)}^{e_n(y)} f\,dxdy\right|}{\left|\int_0^1 \int_{\bar{e}^A}^1 f\,dxdy\right|^2} \leq \frac{\bar{f}\,\|e_n - e\|}{\left|\int_0^1 \int_{\bar{e}^A}^1 f\,dxdy\right|^2}.
\end{aligned}
$$

Thus, for any $n = 1, 2, \ldots$, we have

$$
\|G_n(\cdot|in) - G(\cdot|in)\| \leq \frac{\bar{f}\,\|e_n - e\|}{\left|\int_0^1 \int_{\bar{e}^A}^1 f\,dxdy\right|^2}.
$$

Since $\lim_{n\to\infty} \|e_n - e\| = 0$, we have $\lim_{n\to\infty} \|G_n(\cdot|in) - G(\cdot|in)\| = 0$.

Similarly, for any $n = 1, 2, \ldots$ and any $z \in [0, 1]$, we have

$$
\begin{aligned}
&|G_n(z|out) - G(z|out)| = \left| \frac{\int_0^z \int_0^{e_n(y)} f \, dxdy}{\int_0^1 \int_0^{e_n(y)} f \, dxdy} - \frac{\int_0^z \int_0^{e(y)} f \, dxdy}{\int_0^1 \int_0^{e(y)} f \, dxdy} \right| \\
=& \left| \frac{\int_0^z \int_0^{e_n(y)} f \, dxdy \times \int_0^1 \int_0^{e(y)} f \, dxdy - \int_0^z \int_0^{e(y)} f \, dxdy \times \int_0^1 \int_0^{e_n(y)} f \, dxdy}{\int_0^1 \int_0^{e_n(y)} f \, dxdy \times \int_0^1 \int_0^{e(y)} f \, dxdy} \right| \\
=& \left| \frac{\int_0^z \int_0^{e_n(y)} f \, dxdy \times \int_z^1 \int_0^{e(y)} f \, dxdy - \int_0^z \int_0^{e(y)} f \, dxdy \times \int_z^1 \int_0^{e_n(y)} f \, dxdy}{\int_0^1 \int_0^{e_n(y)} f \, dxdy \times \int_0^1 \int_0^{e(y)} f \, dxdy} \right| \\
=& \frac{1}{\left| \int_0^1 \int_0^{e_n(y)} f \, dxdy \times \int_0^1 \int_0^{e(y)} f \, dxdy \right|} \times \\
& \left| \int_0^z \int_0^{e_n(y)} f \, dxdy \times \int_z^1 \int_0^{e(y)} f \, dxdy - \int_0^z \int_0^{e_n(y)} f \, dxdy \times \int_z^1 \int_0^{e_n(y)} f \, dxdy \right. \\
& \left. + \int_0^z \int_0^{e_n(y)} f \, dxdy \times \int_z^1 \int_0^{e_n(y)} f \, dxdy - \int_0^z \int_0^{e(y)} f \, dxdy \times \int_z^1 \int_0^{e_n(y)} f \, dxdy \right| \\
=& \frac{\left| - \int_0^z \int_0^{e_n(y)} f \, dxdy \times \int_z^1 \int_{e(y)}^{e_n(y)} f \, dxdy + \int_0^z \int_{e(y)}^{e_n(y)} f \, dxdy \times \int_z^1 \int_0^{e_n(y)} f \, dxdy \right|}{\left| \int_0^1 \int_0^{e_n(y)} f \, dxdy \times \int_0^1 \int_0^{e(y)} f \, dxdy \right|} \\
\leq& \frac{\left| \int_0^z \int_0^{e_n(y)} f \, dxdy \right| \left| \int_z^1 \int_{e(y)}^{e_n(y)} f \, dxdy \right| + \left| \int_0^z \int_{e(y)}^{e_n(y)} f \, dxdy \right| \left| \int_z^1 \int_0^{e_n(y)} f \, dxdy \right|}{\left| \int_0^1 \int_0^{e_n(y)} f \, dxdy \right| \left| \int_0^1 \int_0^{e(y)} f \, dxdy \right|} \\
\leq& \frac{\left| \int_z^1 \int_{e(y)}^{e_n(y)} f \, dxdy \right| + \left| \int_0^z \int_{e(y)}^{e_n(y)} f \, dxdy \right|}{\left| \int_0^{r^B + c^B} \int_0^{\underline{e}^A} f \, dxdy \right|^2} \leq \frac{\bar{f} \, \|e_n - e\|}{\left| \int_0^{r^B + c^B} \int_0^{\underline{e}^A} f \, dxdy \right|^2}.
\end{aligned}
$$

Thus, for any $n = 1, 2, \ldots$, we have

$$
\|G_n(\cdot|out) - G(\cdot|out)\| \leq \frac{\bar{f} \, \|e_n - e\|}{\left| \int_0^{r^B + c^B} \int_0^{\underline{e}^A} f \, dxdy \right|^2}.
$$

Since $\lim_{n \to \infty} \|e_n - e\| = 0$, we have $\lim_{n \to \infty} \|G_n(\cdot|out) - G(\cdot|out)\| = 0$.

*Step 2.* The sequence $\{(e_n^{oo}, e_n^{oi}, e_n^{io}, e_n^{ii})\}_{n=1}^{\infty}$ converges to $(e^{oo}, e^{oi}, e^{io}, e^{ii})$.

As in the proof of Proposition 1, we define $\lambda(v^B) = (v^B - r^B) G(v^B|out)$ for all $v^B \in [r^B, 1]$. For all $n = 1, 2, \ldots$, let $\lambda_n(v^B) = (v^B - r^B) G_n(v^B|out)$ for all $v^B \in [r^B, 1]$. Since $G_n(z|out) > 0$, for all $n = 1, 2, \ldots$, and $G(z|out) > 0$ for all $z > 0$, the functions $\lambda_n$, for all $n = 1, 2, \ldots$, and $\lambda$ are increasing on $[r^B, 1]$, and thus their inverses are well-defined with domain $[0, 1 - r^B]$. Since $|\lambda_n(v^B) - \lambda(v^B)| = |v^B - r^B||G_n(v^B|out) - G(v^B|out)|$ for any

$n = 1, 2, \ldots$ and any $v^B \in [r^B, 1]$, we have

$$\|\lambda_n - \lambda\| \leq (1 - r^B)\|G_n(\cdot|out) - G(\cdot|out)\|$$

for any $n = 1, 2, \ldots$. Since $\lim_{n\to\infty} \|G_n(\cdot|out) - G(\cdot|out)\| = 0$, we have $\lim_{n\to\infty} \|\lambda_n - \lambda\| = 0$. Since $\{\lambda_n\}$ converges uniformly to $\lambda$ and $\lambda$ is continuous, $\{\lambda_n^{-1}\}$ converges uniformly to $\lambda^{-1}$ on $[0, 1 - r^B]$ (see Theorem 1 of Barvínek *et al.*, 1991). Note that $c^B \in (0, 1 - r^B)$, $e_n^{oo} = \lambda_n^{-1}(c^B)$, for $n = 1, 2, \ldots$, and $e^{oo} = \lambda^{-1}(c^B)$. Since uniform convergence implies pointwise convergence, $\{e_n^{oo}\}$ converges to $e^{oo}$.

The convergence of $\{e_n^{ii}\}$ to $e^{ii}$ can be shown analogously as above, with $G_n(\cdot|out)$ and $G(\cdot|out)$ replaced by $G_n(\cdot|in)$ and $G(\cdot|in)$, respectively.

Next we show the convergence of $\{(e_n^{oi}, e_n^{io})\}$ to $(e^{oi}, e^{io})$. Recall the definitions of the functions $\phi$ and $\kappa$ in (22) and (23), respectively, and define $\phi_n$ analogously to $\phi$ using $G_n(\cdot|out)$ and $\kappa_n$ analogously to $\kappa$ using $\phi_n$ and $G_n(\cdot|in)$. Note that $\phi$ and $\kappa$ are defined on $[e^{oo}, 1]$ while $\phi_n$ and $\kappa_n$ are defined on $[e_n^{oo}, 1]$, for all $n = 1, 2, \ldots$. Let $\bar{e}^{oo} = \sup\{e^{oo}, e_1^{oo}, e_2^{oo}, \ldots\}$.

For any $n = 1, 2, \ldots$ and any $v \in [\bar{e}^{oo}, 1]$, we have

$$\begin{aligned}
|\phi_n(v) - \phi(v)| &= \left| \frac{c^B}{G_n(v|out)} - \frac{c^B}{G(v|out)} \right| \\
&= \frac{c^B}{G_n(v|out)G(v|out)} \left|G(v|out) - G_n(v|out)\right|
\end{aligned}$$

Since $v \geq \bar{e}^{oo} \geq e^{oo}$, we have $G(v|out) \geq G(e^{oo}|out) = c^B/(e^{oo} - r^B) \geq c^B/(1 - r^B)$, and similarly we have $G_n(v|out) \geq c^B/(1 - r^B)$. Thus, for any $n = 1, 2, \ldots$, we have

$$\|\phi_n - \phi\| \leq \frac{(1 - r^B)^2}{c^B} \left\|G_n(\cdot|out) - G(\cdot|out)\right\|.$$

Since $\lim_{n\to\infty} \|G_n(\cdot|out) - G(\cdot|out)\| = 0$, we have $\lim_{n\to\infty} \|\phi_n - \phi\| = 0$ on $[\bar{e}^{oo}, 1]$.

For any $n = 1, 2, \ldots$ and any $v \in [\bar{e}^{oo}, 1]$, we have

$$
\begin{aligned}
&|\kappa_n(v) - \kappa(v)| \\
=&\left| \frac{c^B}{G_n(v|out)} G_n(\phi_n(v)|in) + \int_{\phi_n(v)}^{v} G_n(y|in)dy - \frac{c^B}{G(v|out)} G(\phi(v)|in) - \int_{\phi(v)}^{v} G(y|in)dy \right| \\
=&\left| \frac{c^B}{G_n(v|out)} G_n(\phi_n(v)|in) - \frac{c^B}{G_n(v|out)} G(\phi(v)|in) + \frac{c^B}{G_n(v|out)} G(\phi(v)|in) \right. \\
&\left. - \frac{c^B}{G(v|out)} G(\phi(v)|in) + \int_{\phi_n(v)}^{\phi(v)} G_n(y|in)dy + \int_{\phi(v)}^{v} [G_n(y|in) - G(y|in)]dy \right| \\
\leq& \frac{c^B}{G_n(v|out)} \left| G_n(\phi_n(v)|in) - G(\phi(v)|in) \right| + G(\phi(v)|in) \left| \frac{c^B}{G_n(v|out)} - \frac{c^B}{G(v|out)} \right| \\
&+ |\phi_n(v) - \phi(v)| + (v - \phi(v)) \left\| G_n(\cdot|in) - G(\cdot|in) \right\| \\
\leq& (1 - r^B) \left| G_n(\phi_n(v)|in) - G(\phi(v)|in) \right| + 2\|\phi_n - \phi\| + (1 - r^B - c^B) \left\| G_n(\cdot|in) - G(\cdot|in) \right\|.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&|G_n(\phi_n(v)|in) - G(\phi(v)|in)| \\
=& |G_n(\phi_n(v)|in) - G(\phi_n(v)|in) + G(\phi_n(v)|in) - G(\phi(v)|in)| \\
\leq& |G_n(\phi_n(v)|in) - G(\phi_n(v)|in)| + |G(\phi_n(v)|in) - G(\phi(v)|in)| \\
\leq& \|G_n(\cdot|in) - G(\cdot|in)\| + \sup_{z \in (0,1)} \left| G'(z|in) \right| |\phi_n(v) - \phi(v)| \\
\leq& \|G_n(\cdot|in) - G(\cdot|in)\| + \frac{\bar{f}}{\int_0^1 \int_0^{\bar{e}^A} f \, dxdy} \|\phi_n - \phi\|
\end{aligned}
$$

Hence, for any $n = 1, 2, \ldots$, we have

$$
\|\kappa_n - \kappa\| \leq [2(1 - r^B) - c^B] \|G_n(\cdot|in) - G(\cdot|in)\| + \left[ 2 + \frac{(1 - r^B)\bar{f}}{\int_0^1 \int_0^{\bar{e}^A} f \, dxdy} \right] \|\phi_n - \phi\|.
$$

Since $\lim_{n\to\infty} \|G_n(\cdot|in) - G(\cdot|in)\| = 0$ and $\lim_{n\to\infty} \|\phi_n - \phi\| = 0$, we have $\lim_{n\to\infty} \|\kappa_n - \kappa\| = 0$ on $[\bar{e}^{oo}, 1]$.

Suppose that $\kappa(1) < c^B$. Then $e^{oi} = 1$ and $e^{io} = r^B + c^B$. Since $\{\kappa_n\}$ converges uniformly to $\kappa$ on $[\bar{e}^{oo}, 1]$, $\{\kappa_n(1)\}$ converges to $\kappa(1)$. Hence, $\kappa_n(1) < c^B$ for sufficiently large $n$, and $e_n^{oi} = 1$ and $e_n^{io} = r^B + c^B$ for sufficiently large $n$. This implies that $\{(e_n^{oi}, e_n^{io})\}$ converges to $(e^{oi}, e^{io})$.

Suppose that $\kappa(1) > c^B$. Recall that $\kappa(e^{oo}) < c^B$. Choose any $\varepsilon \in (0, c^B - \kappa(e^{oo}))$. Since $\kappa$ is continuous at $e^{oo}$, we can find $\delta > 0$ such that $\kappa(e^{oo} + \delta) < \kappa(e^{oo}) + \varepsilon$. Since $\{e_n^{oo}\}$ converges to $e^{oo}$, we have $e_n^{oo} < e^{oo} + \delta$ for sufficiently large $n$. So without loss of generality we assume that $\bar{e}^{oo} \leq e^{oo} + \delta$. Since $\{\kappa_n\}$ converges uniformly to $\kappa$ on $[\bar{e}^{oo}, 1]$, we have $\kappa_n(e^{oo}+\delta) < \kappa(e^{oo})+\varepsilon$ and $\kappa_n(1) > c^B$ for sufficiently large $n$. That is, for sufficiently large $n$, $\kappa_n$ and $\kappa$ are defined on $[e^{oo}+\delta, 1]$ with $[\kappa(e^{oo})+\varepsilon, c^B]$ included in $\kappa_n([e^{oo}+\delta, 1])$ and $\kappa([e^{oo}+\delta, 1])$, and we focus on such large $n$. Since $\kappa_n$ and $\kappa$ are increasing, their inverses are well-defined on $[\kappa(e^{oo}) + \varepsilon, c^B]$. Since $\kappa$ is continuous, $\{\kappa_n^{-1}\}$ converges uniformly to $\kappa^{-1}$ on $[\kappa(e^{oo}) + \varepsilon, c^B]$ . Note that $e_n^{oi} = \kappa_n^{-1}(c^B)$, for all $n = 1, 2, \ldots$, and $e^{oi} = \kappa^{-1}(c^B)$. Since uniform convergence implies pointwise convergence, $\{e_n^{oi}\}$ converges to $e^{oi}$. Also, note that $e_n^{io} = \phi_n(e_n^{oi})$, for all $n = 1, 2, \ldots$, and $e^{io} = \phi(e^{oi})$. We have $e^{oi} > e^{oo} + \delta \geq \bar{e}^{oo}$ and $e_n^{oi} > e^{oo} + \delta \geq \bar{e}^{oo}$ for sufficiently large $n$. Since $\{\phi_n\}$ converges uniformly to continuous $\phi$ on $[\bar{e}^{oo}, 1]$ and $\{e_n^{oi}\}$ converges to $e^{oi}$, $\{e_n^{io}\}$ converges to $e^{io}$ (see Theorem 1 of Kolk, 1999).

Lastly, suppose that $\kappa(1) = c^B$. Then we have $e^{oi} = 1$ and $e^{io} = r^B + c^B$. For any $n$ such that $\kappa_n(1) \leq c^B$, we have $e_n^{oi} = 1$ and $e_n^{io} = r^B + c^B$. Suppose that $\kappa_n(1) > c^B$ for finitely many $n$. Then we have $e_n^{oi} = 1$ and $e_n^{io} = r^B + c^B$ for all $n$ larger than some finite number, and the result follows. Suppose that $\kappa_n(1) > c^B$ for infinitely many $n$. Then we can focus on the subsequence of $\{\kappa_n\}$ such that $\kappa_n(1) > c^B$ and apply the same argument as in the previous paragraph to complete the proof.

*Step 3.* The sequences $\{\pi_n^A\}_{n=1}^\infty$ and $\{\Pi_n^B\}_{n=1}^\infty$ converge uniformly to $\pi^A$ and $\Pi^B$, respectively.

For any $n = 1, 2, \ldots$ and any $v^A \in [r^A, 1]$, we have

$$\begin{aligned}
&\left|\pi_n^A(v^A) - \pi^A(v^A)\right| \\
=&\left|\int_0^1 \int_0^{e_n(y)} (v^A - r^A) f \, dxdy + \int_0^1 \int_{\min\{e_n(y), v^A\}}^{v^A} (v^A - x) f \, dxdy \right. \\
&\left. - \int_0^1 \int_0^{e(y)} (v^A - r^A) f \, dxdy - \int_0^1 \int_{\min\{e(y), v^A\}}^{v^A} (v^A - x) f \, dxdy \right| \\
=&\left|\int_0^1 \int_{e(y)}^{e_n(y)} (v^A - r^A) f \, dxdy + \int_0^1 \int_{\min\{e_n(y), v^A\}}^{\min\{e(y), v^A\}} (v^A - x) f \, dxdy \right| \\
\leq&(v^A - r^A)\bar{f} \left( \int_0^1 |e_n(y) - e(y)| \, dy + \int_0^1 \left|\min\{e_n(y), v^A\} - \min\{e(y), v^A\}\right| dy \right).
\end{aligned}$$

Since $\left|\min\{e_n(y), v^A\} - \min\{e(y), v^A\}\right| \leq |e_n(y) - e(y)|$ for any $y \in [0, 1]$ and any $v^A \in [r^A, 1]$, we have for any $n = 1, 2, \ldots$

$$\left\|\pi_n^A - \pi^A\right\| \leq 2(1 - r^A)\bar{f} \left\|e_n - e\right\|.$$

Since $\lim_{n\to\infty} \|e_n - e\| = 0$, we have $\lim_{n\to\infty} \left\|\pi_n^A - \pi^A\right\| = 0$.

Suppose that $v^B < e^{io}$. Then for sufficiently large $n$, we have $v^B < e_n^{io}$ and $\left|\Pi_n^B(v^B) - \Pi^B(v^B)\right| =$

0. Suppose that $e^{io} < v^B < e^{oo}$. Then for sufficiently large $n$, we have $e_n^{io} < v^B < e_n^{oo}$ and

$$
\begin{aligned}
&\left|\Pi_n^B(v^B) - \Pi^B(v^B)\right| \\
=&\left|\int_0^{e_n^{oi}} \int_0^{e_n(y)} (v^B - r^B) f\, dxdy - c^B \int_0^1 \int_0^{e_n(y)} f\, dxdy \right.\\
&\left. - \int_0^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f\, dxdy + c^B \int_0^1 \int_0^{e(y)} f\, dxdy \right| \\
=&\left|\int_0^{e_n^{oi}} \int_{e(y)}^{e_n(y)} (v^B - r^B) f\, dxdy - \int_{e_n^{oi}}^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f\, dxdy - c^B \int_0^1 \int_{e(y)}^{e_n(y)} f\, dxdy \right| \\
\leq& \bar{f}\left[(1 + c^B)\left\|e_n - e\right\| + \left|e_n^{oi} - e^{oi}\right|\right].
\end{aligned}
$$

Suppose that $e^{oo} < v^B < e^{ii}$. Then for sufficiently large $n$, we have $e_n^{oo} < v^B < e_n^{ii}$ and

$$
\begin{aligned}
&\left|\Pi_n^B(v^B) - \Pi^B(v^B)\right| \\
=&\left|\int_{v^B}^{e_n^{oi}} \int_0^{e_n(y)} (v^B - r^B) f\, dxdy + \int_{e_n^{oo}}^{v^B} \int_0^{e_n(y)} (y - r^B) f\, dxdy \right.\\
&\left. - \int_{v^B}^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f\, dxdy - \int_{e^{oo}}^{v^B} \int_0^{e(y)} (y - r^B) f\, dxdy \right| \\
=&\left|\int_{v^B}^{e_n^{oi}} \int_{e(y)}^{e_n(y)} (v^B - r^B) f\, dxdy - \int_{e_n^{oi}}^{e^{oi}} \int_0^{e(y)} (v^B - r^B) f\, dxdy \right.\\
&\left. + \int_{e_n^{oo}}^{e^{oo}} \int_0^{e_n(y)} (y - r^B) f\, dxdy - \int_{e^{oo}}^{v^B} \int_{e_n(y)}^{e(y)} (y - r^B) f\, dxdy \right| \\
\leq& \bar{f}\left(2\left\|e_n - e\right\| + \left|e_n^{oi} - e^{oi}\right| + \left|e_n^{oo} - e^{oo}\right|\right).
\end{aligned}
$$

Suppose that $e^{ii} < v^B < e^{oi}$. Then for sufficiently large $n$, we have $e_n^{ii} < v^B < e_n^{oi}$ and

$$
\begin{aligned}
&\left|\Pi_n^B(v^B) - \Pi^B(v^B)\right| \\
\leq& \bar{f}\left[(4 + c^B)\left\|e_n - e\right\| + \left|e_n^{oi} - e^{oi}\right| + \left|e_n^{oo} - e^{oo}\right| + 2\left|e_n^{ii} - e^{ii}\right|\right].
\end{aligned}
$$

Suppose that $v^B > e^{oi}$. Then for sufficiently large $n$, we have $v^B > e_n^{oi}$ and

$$\begin{aligned}
&\left|\Pi_n^B(v^B) - \Pi^B(v^B)\right| \\
=&\left|\int_{e_n^{oo}}^{e_n^{oi}} \int_0^{e_n(y)} (y - r^B) f \, dxdy + \int_{e_n^{io}}^{e_n^{ii}} \int_{e_n(y)}^1 (y - r^B) f \, dxdy \right.\\
&\left. - \int_{e^{oo}}^{e^{oi}} \int_0^{e(y)} (y - r^B) f \, dxdy - \int_{e^{io}}^{e^{ii}} \int_{e(y)}^1 (y - r^B) f \, dxdy \right| \\
=&\left|\int_{e_n^{oo}}^{e_n^{oi}} \int_{e(y)}^{e_n(y)} (y - r^B) f \, dxdy - \int_{e^{oo}}^{e_n^{oo}} \int_0^{e(y)} (y - r^B) f \, dxdy - \int_{e_n^{oi}}^{e^{oi}} \int_0^{e(y)} (y - r^B) f \, dxdy \right.\\
&\left. + \int_{e_n^{io}}^{e_n^{ii}} \int_{e_n(y)}^{e(y)} (y - r^B) f \, dxdy - \int_{e^{io}}^{e_n^{io}} \int_{e(y)}^1 (y - r^B) f \, dxdy - \int_{e_n^{ii}}^{e^{ii}} \int_{e(y)}^1 (y - r^B) f \, dxdy \right| \\
\le&\bar{f} \left(2 \|e_n - e\| + |e_n^{oo} - e^{oo}| + \left|e_n^{oi} - e^{oi}\right| + \left|e_n^{io} - e^{io}\right| + \left|e_n^{ii} - e^{ii}\right|\right).
\end{aligned}$$

Suppose that $v^B \in \{e^{io}, e^{oo}, e^{ii}, e^{oi}\}$. Since $\Pi^B$ is continuous, the expression for $\Pi^B(v^B)$ can be chosen to the corresponding one for $\Pi_n^B(v^B)$ for sufficiently large $n$, and we can obtain an upper bound of $\left|\Pi_n^B(v^B) - \Pi^B(v^B)\right|$ as above. To sum up, for sufficiently large $n$, we have

$$\begin{aligned}
&\left\|\Pi_n^B - \Pi^B\right\| \\
\le&\bar{f} \max\left\{(4 + c^B) \|e_n - e\| + \left|e_n^{oi} - e^{oi}\right| + |e_n^{oo} - e^{oo}| + 2\left|e_n^{ii} - e^{ii}\right|,\right.\\
&\qquad\left. 2\|e_n - e\| + |e_n^{oo} - e^{oo}| + \left|e_n^{oi} - e^{oi}\right| + \left|e_n^{io} - e^{io}\right| + \left|e_n^{ii} - e^{ii}\right|\right\}.
\end{aligned}$$

Since $\lim_{n\to\infty} \|e_n - e\| = 0$, $\lim_{n\to\infty} |e_n^{oo} - e^{oo}| = 0$, $\lim_{n\to\infty} \left|e_n^{oi} - e^{oi}\right| = 0$, $\lim_{n\to\infty} \left|e_n^{io} - e^{io}\right| = 0$, and $\lim_{n\to\infty} \left|e_n^{ii} - e^{ii}\right| = 0$, we have $\lim_{n\to\infty} \left\|\Pi_n^B - \Pi^B\right\| = 0$.

*Step 4.* The sequence $\{Te_n\}_{n=1}^{\infty}$ converges uniformly to $Te$.

Recall that $\pi_n^A$, for all $n = 1, 2, \ldots$, and $\pi^A$ are increasing on $[r^A, 1]$, and their inverses are well-defined on $[-c^A, 0]$. Since $\{\pi_n^A\}$ converges uniformly to $\pi^A$ and $\pi^A$ is continuous, $\{(\pi_n^A)^{-1}\}$ converges uniformly to $(\pi^A)^{-1}$ on $[-c^A, 0]$. Also, note that

$$\left|\frac{d(\pi^A)^{-1}(z)}{dz}\right| = \left|\frac{1}{(\pi_A)'((\pi^A)^{-1}(z))}\right| \le L$$

for all $z \in (-c^A, 0)$. From (29), we obtain $Te(v^B) = (\pi^A)^{-1}(-\Pi^B(v^B))$ and $Te_n(v^B) = (\pi_n^A)^{-1}(-\Pi_n^B(v^B))$, for all $n = 1, 2, \ldots$, for all $v^B \in [0, 1]$. For any $n = 1, 2, \ldots$ and any

$v^B \in [0,1]$, we have

$$\begin{aligned}
&\left|Te_n(v^B) - Te(v^B)\right| \\
=&\left|(\pi_n^A)^{-1}(-\Pi_n^B(v^B)) - (\pi^A)^{-1}(-\Pi^B(v^B))\right| \\
=&\left|(\pi_n^A)^{-1}(-\Pi_n^B(v^B)) - (\pi^A)^{-1}(-\Pi_n^B(v^B)) + (\pi^A)^{-1}(-\Pi_n^B(v^B)) - (\pi^A)^{-1}(-\Pi^B(v^B))\right| \\
\leq&\left|(\pi_n^A)^{-1}(-\Pi_n^B(v^B)) - (\pi^A)^{-1}(-\Pi_n^B(v^B))\right| + \left|(\pi^A)^{-1}(-\Pi_n^B(v^B)) - (\pi^A)^{-1}(-\Pi^B(v^B))\right| \\
\leq&\left\|(\pi_n^A)^{-1} - (\pi^A)^{-1}\right\| + L\left\|\Pi_n^B - \Pi^B\right\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\left\|(\pi_n^A)^{-1} - (\pi^A)^{-1}\right\| = 0$ and $\lim_{n\to\infty}\left\|\Pi_n^B - \Pi^B\right\| = 0$, we have $\lim_{n\to\infty}\|Te_n - Te\| = 0$. □

**Proof of Proposition 4:** Since $e$ is monotonic, it is continuous almost everywhere, and thus Proposition 2 applies. In particular, $e$ is constant on $[0, e^{io}]$ with $e(0) > r^A + c^A$, and $e(v^B) < 1$ for all $v^B \in [e^{ii}, 1]$. Then following the proof of Proposition 1, we can show that $G(z|in) < G(z|out)$ for all $z \in (0,1)$.

The equilibrium cutoff $e^B$ is determined by (4) with $k = B$. Comparing this with the conditions (18) and (19) for $e^{oo}$ and $e^{ii}$, we obtain $e^{oo} < e^B < e^{ii}$ once we establish that $G(z|in) < F_B(z) < G(z|out)$ for all $z \in (0,1)$. Fix any $z \in (0,1)$. Note that

$$F_B(z) = \int_0^z \int_0^1 f(x,y)dxdy = \frac{\int_0^z \int_0^{e(y)} f(x,y)dxdy + \int_0^z \int_{e(y)}^1 f(x,y)dxdy}{\int_0^1 \int_0^{e(y)} f(x,y)dxdy + \int_0^1 \int_{e(y)}^1 f(x,y)dxdy}.$$

Since $a/b < c/d$ implies $a/b < (a+c)/(b+d) < c/d$ for any $a \geq 0$ and $b,c,d > 0$, the relationships $G(z|in) < F_B(z) < G(z|out)$ follow from $G(z|in) < G(z|out)$.

The equilibrium cutoff $e^A$ is determined by (4) with $k = A$. Note that $r^A + c^A < e^A < 1$. So to show $e^A > e(1)$, it suffices to consider the case where $e(1) > r^A + c^A$. Since $e$ is nonincreasing and $\Pi^B(1) > 0$, we have

$$\begin{aligned}
0 &= \pi^A(e(1)) + \Pi^B(1) \\
&= \int_0^1 \int_0^{e(y)} (e(1) - r^A) f(x,y)dxdy - c^A + \Pi^B(1) \\
&> \int_0^1 \int_0^{e(1)} (e(1) - r^A) f(x,y)dxdy - c^A = (e(1) - r^A)F_A(e(1)) - c^A.
\end{aligned}$$

Since $(x - r^A)F_A(x)$ is increasing in $x$ on $[r^A, 1]$, it follows that $e^A > e(1)$. Next, to show $e^A < e(0)$, it suffices to consider the case where $e(0) < 1$. Recall that $\tilde{e}(v^B) = \max\{e(v^B), r^A\}$ for all $v^B \in [0,1]$. Since $\tilde{e}(v^B) \geq r^A$ for all $v^B \in [0,1]$, $e(0) > r^A + c^A$, and $e$ is decreasing on some non-degenerate interval, we have

$$
\begin{aligned}
0 &= \pi^A(e(0)) + \Pi^B(0) \\
&= \int_0^1 \int_0^{\tilde{e}(y)} (e(0) - r^A) f(x,y)dxdy + \int_0^1 \int_{\tilde{e}(y)}^{e(0)} (e(0) - x) f(x,y)dxdy - c^A \\
&= \int_0^1 \int_0^{e(0)} (e(0) - r^A) f(x,y)dxdy - \int_0^1 \int_{\tilde{e}(y)}^{e(0)} (x - r^A) f(x,y)dxdy - c^A \\
&< \int_0^1 \int_0^{e(0)} (e(0) - r^A) f(x,y)dxdy - c^A = (e(0) - r^A)F_A(e(0)) - c^A.
\end{aligned}
$$

It follows that $e^A < e(0)$. $\square$

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