On the (Robust) Ex-post Stability of Constitutions*

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Abstract

Barbera and Jackson (2004) define a constitution as a pair of voting rules (f, F), where f is employed for ordinary decisions, and F is employed to choose between f and a proposed voting rule. While they study the stability of constitutions at the ex-ante stage, where agents' preferences over final outcomes are uncertain, we focus on the ex-post stage, where agents' preferences are known. We present a characterization of ex-post stable constitutions. Furthermore, we examine the robustness of this characterization to the changes in the voting environment and the relationship between ex-post stability and ex-ante stability of constitutions.

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1 Introduction

Voting rules play a crucial role in decision-making for groups. In a group, agents' preferences depend on the final outcomes. Since different rules can result in significantly different outcomes, the agents' preferences naturally translate into preferences over voting rules. It is important to understand the type of voting rules that would survive in the long run, given the chance to change the prevailing voting rule and how would changes in the voting environment impact this decision. Additionally, it is crucial to consider how this decision might differ based on the amount of information that agents have about each other's preferences.

To address these question, we employ the theoretical framework presented by Barbera and Jackson (2004). They define a constitution as a pair of voting rules (f, F) where the given rule f is employed for ordinary decisions and the base rule F is used to choose between f and a proposed voting rule. They assume an uncertain situation regarding the preferences of all voters when the voting rule is chosen. They propose a definition of an ex-ante stable constitutions[1], which is a constitution(f, F) that supports the existing rule f against any alternative rule under F. They examined specific cases of ex-ante stable constitutions, like when F is identical to f (self-stable voting rule), or when f is a simple majority rule or F is a unanimity rule. They also investigated how the voting environment affects ex-ante stability in these cases. However, the complete characterization of ex-ante stable constitutions remains unclear.

This paper is to shift the analysis of stable constitutions from an ex-ante perspective, where uncertainty about all agents' preferences exists, to an ex-post perspective, where the uncertainty is resolved. This approach of analyzing based on the amount of information at the point of choice is a conventional approach in theoretical economics. Holmstrom and Myerson (1983) defined Pareto efficiency from ex-ante, interim, and ex-post viewpoints. For the study of stable constitutions, Jeong and Kim (2022) consider the interim perspective, where agents know their own preferences but are uncertain about the preferences of others. We aim to fill a theoretical gap in the literature by employing the ex-post perspective. In reality, the choice of voting rules at the ex-post stage can be observed in various contexts. For instance, the preferences of left and right are often known to everyone in political

[1] Barbera and Jackson (2004) refer to it as a self-stable constitution. To emphasize the current ex-post approach and various base rules, this paper refers to it as an ex-ante stable constitutions.

situations. In small communities or organizations, where individuals know each other well, or in situations where there have been intense discussions about the next issue, preferences for the issue may be known to each other. Furthermore, for myopic agents who are more focused on the next issue than future uncertain issues, the ex-post perspective may be more appropriate.

The primary finding of this paper is a complete characterization of ex-post stable constitutions. Furthermore, we investigate the robustness of the set of ex-post stable constitutions in response to variations in the voting environment, where the voting environment refers to the set of competing alternative rules. We find a sufficient condition that readily met in practice to maintain the characterization. Additionally, we discuss the relationship between ex-post stability and ex-ante stability of constitutions, emphasizing the differences and similarities between these concepts.

Specifically, we define the ex-post stable constitution as a pair of voting rules (f, F) that support f over any other rule g under F in all cases where agents' preferences (called types) are known. We demonstrate that a constitution (f, F) is ex-post stable if and only if every winning coalition of the base rule F is both the winning and veto coalition of the given rule f (Proposition 1). A winning coalition is a group of agents who has sufficient voting power to choose a proposal. A veto coalition is a group of agents who possess sufficient voting power to reject a proposal. Intuitively, in the ex-post stable constitutions, agents who can choose a proposed voting rule under F must possess enough voting power in f to determine their desired outcome for both of their types. In addition, we discover that the winning coalition of the base rule F must be the veto coalition of F (Corollary 1). This provides information about the range of the base rule. For example, the submajority rule cannot be a base rule F in any ex-post stable constitution.

Proposition 2 investigates the influence of changes in the voting environment on the set of ex-post stable constitutions. While Proposition 1 assumes a fixed set of general voting rules as the voting environment, Proposition 2 presents a sufficient condition to maintain the characterization identified in Proposition 1 even when the set of competing alternative rules varies. This condition requires that the set of alternative rules includes a rule where every winning coalition is also a veto coalition. This condition is easily satisfied in practice as commonly employed voting rules such as majority rule and unanimity rule meet this requirement. The observation that the set of ex-post stable constitutions is robust to changes in the set of alternative rules, contrasts with the set of ex-ante and interim stable constitutions.

Proposition 3 shows that if the set of competing alternative rules consists of general voting rules, then the ex-ante stable constitution is also ex-post stable. However, this relationship does not hold when the set of alternative rules is limited to anonymous voting rules, as demonstrated by several examples in Barbera and Jackson (2004).

We now discuss how primary findings of this paper are related to previous literature. Barbera and Jackson (2004) solely focused on anonymous voting rules, whereas Azrieli and Kim (2016) extended this approach by including weighted majority rules. Azrieli and Kim (2016) characterized ex-ante self-stable weighted majority rules. Jeong and Kim (2022) obtained a set of interim stable constitutions, comprising anonymous rules, competing against the set of weighted majority rules. From an interim perspective, Holmstrom and Myerson (1983) introduced the notion of durability. The main difference between durability and interim stability is that F is limited to the unanimity rule, while f is not confined to a binary decision rule for durability. However, none of these studies fully characterized stable constitutions from their respective perspectives. This paper expands on these previous studies by offering a complete characterization of ex-post stable constitutions, which may serve as a basis for future research on the complete characterization of ex-ante or interim stable constitutions.

The set of ex-ante stable constitutions depends on the set of competing alternative rules. This leads to variations in dependence of the set ex-ante stable constitutions on voting environments, such as the probability of agents favoring reforms. Barbera and Jackson (2004) demonstrated that when the competing rules are anonymous, the set is dependent on the probability. In Azrieli and Kim (2016), where the competing rules are arbitrary randomized rules, the set is independent of the probability. However, this paper demonstrates a high degree of robustness of ex-post stable constitutions to changes in the set of alternative rules.

Furthermore, Jeong and Kim (2022) employed a slightly different definition of ex-post stability and show that interim stable constitutions are ex-post stable confined to anonymous rules. This paper presents the relationship between ex-ante and ex-post stable constitutions.

2 The Model

2.1 Environment

Society is presented with a binary decision to either implement a Reform (R) or maintain the Status Quo (S). Let $A = \{R, S\}$ be the set of outcomes. There are $n \geq 2$ agents in the society indexed by $i \in N = \{1, \ldots, n\}$.

Each agent can either prefer R or S, indicating a type of agent denoted by $t_i \in \{r, s\} = T_i$. That is, if $t_i = r$ (*respectively*, s), we say that agent i's type is r (*respectively*, s) and prefers R (*respectively*, S). We denote $T = T_1 \times \cdots \times T_n$ the set of type profiles. An agent's utility depends on the chosen outcome and her own type, $u : A \times T_i \to \mathbb{R}$. We normalized the utility such that $u(S, r) = u(S, s) = 0$, $u(R, r) = 1$, and $u(R, s) = -1$.

A voting rule is employed to aggregate the preferences of agents into a decision. Formally, a voting rule is any mapping $f : 2^N \to A$ with a coalition $C \in 2^N$. Let G represent the set of voting rules. We can divide the power set into the set of winning coalitions and non-winning coalitions since f is deterministic. Given $f \in G$, C is a winning coalition if $f(C) = R$ when all agents in C prefer R and the others prefer S. We denote W^f as the set of winning coalitions for f. For a non-winning coalition, by definition, $f(C) = S$ if $C \notin W^f$.

Given a voting rule f, the ex-post utility of agent i at $t \in T$, is given by

$$U_i(f \mid t) = u(f, t_i)$$

2.2 Constitutions and Ex-post stability

We define a constitution as a pair of voting rules $(f, F) \in (G \times G)$.

Definition 1. A constitution is a pair of voting rules (f, F), where the given rule f is for the choice of final outcomes, reform R or status quo S, and the base rule F is for the choice of rules, the given rule f or an alternative rule $g \in G$.

The base rule is also a voting rule, $F \in G$ but it selects either the alternative rule g (reform) or the given rule f (status quo).[2] Therefore, we can regard it as $F : 2^N \to \{g, f\}$. Let $C(g, f \mid t) = \{i \in N : U_i(g \mid t) > U_i(f \mid t)\}$ be the coalition of agents for which rule g yields a strictly higher ex-post utility than rule f. Similarly, we can define a winning

[2]To simplify the presentation, we slightly abuse the notations

coalition of F, i.e., $C \in W^F$ if and only if $F(C) = g$ when $C = C(g, f \mid t)$. Based on these definitions, we can now define the ex-post stability of constitutions.

Definition 2. A constitution (f, F) is ex-post stale in G if $C(g, f \mid t) \notin W^F$ for any voting rule $g \in G$ at every type profile $t \in T$.

In words, if a constitution (f, F) is ex-post stable in G, no alternative rule $g \in G$ would have the sufficient support to replace f under the base rule F at every type profile $t \in T$.

3 Ex-post stable constitutions

3.1 Characterization

The following Proposition 1 characterizes the set of ex-post stable constitutions.

Proposition 1. *A constitution (f, F) is ex-post stable in G if and only if for every coalition $C \in W^F$, $C \in W^f$ and $N/C \notin W^f$.*

Proof. ($\Rightarrow$)'Only if' direction

1) If there exists a coalition $C \in W^F$ such that $C \notin W^f$, then there exists $t \in T$ such that $f(C) = S$. We can construct an alternative rule $g(C) = R$. For every agent i in C, $U_i(g \mid t) > U_i(f \mid t)$. These agents can replace f with $g \in G$ at t under F which implies that the constitution (f, F) is not ex-post stable in G.

2) If there exists a coalition $C \in W^F$ such that $N/C \in W^f$, then there exists $t \in T$ such that $f(N/C) = R$. We can construct an alternative rule $g(N/C) = S$. For every agent i in C at t which is s-type, $U_i(g \mid t) > U_i(f \mid t)$. These agents can replace f with $g \in G$ at t under F which implies that the constitution (f, F) is not ex-post stable in G.

Before presenting the proof of the other direction, the following lemma is useful to identify the types of agents from the difference in utility between the two rules (a "utility gap").

Lemma 1. *Given a type profile $t \in T$, $f \neq g$ if and only if for each agent $i, j \in N$, $(U_i(g \mid t) - U_i(f \mid t)) \cdot (U_j(g \mid t) - U_j(f \mid t)) > 0$ (respectively, < 0) when $t_i = t_j$ (respectively, $t_i \neq t_j$).*

By definition, the proof is trivial, so omitted. Lemma 1 states that if two voting rules, f and g, are different at a particular type profile t, then every agent will have a nonzero utility gap. Furthermore, if agents have the same type, then the signs of these utility gaps will be the same. However, if the types of agents are different, the signs of the utility gaps will be opposite.

($\Leftarrow$) 'If' direction

Suppose a constitution (f, F) is not ex-post stable in G. Then, there exist an alternative rule $g \in G$ and type profile $t \in T$ such that $F(C(g, f \mid t)) = g$. Since every agent i in $C(g, f \mid t)$ has $U_i(g \mid t) > U_i(f \mid t)$, by Lemma 1, only those agents have the same type. We have the following two cases for the types of those agents.

1) r-type: It must be that $g(C(g, f \mid t)) = R$ and $f(C(g, f \mid t)) = S$. Therefore, $C(g, f \mid t) \notin W^f$

2) s-type: It must be that $g(N/C(g, f \mid t)) = S$ and $f(N/C(g, f \mid t)) = R$. Therefore, $N/C(g, f \mid t) \in W^f$.$\square$

Proposition 1 states that in an ex-post stable constitution (f, F), any winning coalition for F must be both a winning coalition and a veto coalition for f. It is intuitive that agents with both types have enough voting power to achieve their desired outcome through f, and thus have no incentive to replace f with any $g \in G$ under F. Also, if the winning coalition for F is neither a winning coalition nor a veto coalition for f, we can find an alternative rule g supported by those agents.

Let us analyze the ex-post stable constitutions of commonly used rules, using Proposition 1. The unanimity and simple majority rules have been widely employed in practical applications and have received considerable attention in the literature. The following example uses Proposition 1 to derive the set of ex-post constitutions for these cases

Example 1. If the base rule F is a unanimity rule, the constitution (f, F) where any rule f such that $f(N) = R$ and $f(\phi) = S$ is ex-post stable. Additionally, if F is a simple majority rule and n is odd, the constitution $(simple\, majority, simple\, majority)$ is the unique ex-post stable constitution.

Due to Proposition 1, given the base rule F, we can identify the set of rules f for ex-post stable constitutions (f, F). Before we may turn to the question of F for the ex-post stable constitutions, we mention that a base rule with a winning coalition that is not a veto coalition can be problematic in the following sense.[3] Consider a situation where f is the status quo, g is suggested as a reform, $C \in W^F$, $N/C \in W^F$, and $C(g, f \mid t) = C$. Then, g is chosen by F and becomes a new status quo. However, if the previous f is suggested as a reform, then $F(N/C) = f$ since $N/C \in W^F$. Therefore, there is the potential to continuously cycle back and forth between f and g, which could be a serious concern for the long-term stability. Should we exclude the rules with this potential problem when selecting base rules? To address this, we introduce the following concept of a voting rule.

Definition 3. A voting rule f is cycle-proof if every winning coalition of f is a veto coalition, i.e., for any $C \in W^f$, $N/C \notin W^f$.

The following corollary relieves us regarding the above issue of F in ex-post stable constitution (f, F)

Corollary 1. *If a constitution (f, F) is ex-post stable in G, F is cycle-proof.*

Proof. Suppose that a constitution (f, F) is ex-post stable in G, but F is not cycle-proof, meaning that there is a coalition such that $C \in W^F$ and $N/C \in W^F$. According to Proposition 1, for the winning coalition $N/C \in W^F$, $N/C \in W^f$, which contradicts the characterization of an ex-post stable constitution (f, F) in Proposition 1. $\square$

When the base rule $F = f$ and the constitution (f, f) is ex-post stable, such a rule is called an ex-post self-stable rule. Using Proposition 1, we find the ex-post self stable-rules, which turns out to be cycle-proof rules.

Corollary 2. *A rule f is ex-post self-stable if and only if it is cycle-proof.*

The proof is trivial using Proposition 1, so omitted. The concept of cycle-proofness is closely related to stability, which will be further demonstrated in the following subsection.

3.2 Robustness to the set of alternative rules

In the previous subsection, we fixed the set of rules comprising a constitution, f and F, and competing rules g was fixed as the voting rule G. However, the rules of interest may differ and there may be constraints on the competing rules as well. For instance, Barbera and

[3] Barbera and Jackson (2004) point out the potential problem of submajority rules that have a winning coalition which is not a veto coalition.

Jackson (2004) only considered anonymous deterministic rules. Azrieli and Kim (2016) fixed weighted majority rules as the set of f and F but considered the set of arbitrary randomized rules as the set of alternative rules. This subsection explores how changes in the set of alternative rules may affect the set of ex-post stable constitutions in Proposition 1.

Do we have a rule that always belongs to the set of alternative rules? In our setup, the base rule F may be considered an alternative rule since it is employed in the given constitution (f, F). If this point is acceptable, the following lemma holds.

Lemma 2. *Assume given $F \in G'$ for any $G' \subseteq G$. If a constitution (f, F) is ex-post stable in G', then for any $C \in W^F$, $C \in W^f$.*

In words, for any set of alternative rules G' including the base rule F, any winning coalition for F must be a winning coalition for f in the ex-post stable constitution in G'. It is straightforward since the r-type winning coalition for F has the incentive to replace f with F if $f(C) = S$.

Given Lemma 2, can we expect the change in the set of ex-post stable constitutions in response to the change of G'? Due to the competition between the given rule and the alternative rule, it may be a natural prediction that as the set of alternative rules increases, the set of ex-post stable constitutions decreases, and vice versa. However, we can identify a sufficient condition regarding G' that maintains the characterization of ex-post stable constitutions in Proposition 1 and that guarantees that F in any ex-post stable constitution must be cycle-proof.

Proposition 2. *If there exists a cycle-proof rule $g \in G' \subseteq G$, then*

(i) If a constitution (f, F) is ex-post stable in G', F is cycle-proof.

(ii) A constitution (f, F) is ex-post stable in G' if and only if for any $C \in W^F$, $C \in W^f$ and $N/C \notin W^f$.

Proof.

Statement (i): Suppose that there exists a coalition C such that $C \in W^F$ and $N/C \in W^F$. By Lemma 2, $C \in W^f$ and $N/C \in W^f$. We consider the following two cases.

1) $C \in W^g$: Since g is cycle-proof, $g(N/C) = S$ but $f(N/C) = R$ then the coalition C wants to replace f with g , which contradicts that (f, F) is ex-post stable.

2) $C \notin W^g$: $g(C) = S$ but $f(C) = R$, then the coalition N/C wants to replace f with g , which contradicts that (f, F) is ex-post stable.

Statement (ii): It is sufficient to find an alternative rule $g' \in G'$ instead of $g \in G$ in the 'only if' part of the proof of Proposition 1 since other parts remain the same with the exception of replacing G with G'. Since $F(C) = R$ and $F(N/C) = S$ by Statement (i), F can be the alternative rule g'.

The sufficient condition that there exists a cycle-proof rule in G' also ensures that F must be cycle-proof in any ex-post stable constitution. Given that the simple majority or unanimity rule is cycle-proof and the most commonly used for the base rule, it is highly probable that the sufficient condition will be met. The following example shows a case of anonymous rules as the alternative rules, which is widely explored in the literature.

Example 2. The set of ex-post stable constitutions in Proposition 1 remains unaltered when G' is the set of anonymous rules,. This is because the unanimity rule or simple majority rule is cycle-proof and employed an alternative rule. Furthermore, F cannot be a submajority rule in any ex-post stable constitutions (f, F).

4 Relationship with Ex-ante stable constitutions

To examine the relationship with ex-ante stable constitutions, we need additional notations. The ex-ante probability that agent i prefers R is $p_i \in (0, 1)$, and with the complement probability $1 - p_i$ he prefers S, assuming that agents' types are independent. The ex-ante expected utility of agent i for f is denoted by $EU_i(f)$ and is easily computed using the probability. We can define ex-ante stability of constitution with $C(g, f) = \{i \in N : EU_i(g) > EU_i(f)\}$.

Definition 4. A constitution (f, F) is ex-ante stable in G if $C(g, f) \notin W^F$ for any voting rule $g \in G$.

Unlike ex-post stability, the ex-ante stability heavily depends on the set of alternative rules. Barbera and Jackson (2004) considered anonymous rules as the set alternative rules G' and demonstrated that the submajority rule can be ex-ante self-stable in G' and that there could be no self-stable rule, depending on the probability of agents' preference.

However, Proposition 2 implies that a constitution (*submajority*, *submajority*) is not ex-post stable in G' and there is a nonempty set of ex-post stable constitutions. Therefore, it appears that there is no direct relationship between ex-ante stability and ex-post stability of constitutions. However, the following proposition demonstrates that given the set of voting rules G as the set of alternative rule, ex-ante stability implies the ex-post stability of constitutions.

Proposition 3. *If a constitution (f, F) is ex-ante stable in G, it is ex-post stable in G.*

Proof. Suppose that a constitution (f, F) is not ex-post stable in G. There exist the alternative rule $g \in G$ and type profile $t \in T$ such that $F\left(C\left(g, f \mid t\right)\right) = g$. Construct an alternative rule $g'\left(C\right) = g\left(C\right)$ when $C = C\left(g, f \mid t\right)$ and $g'\left(C\right) = f\left(C\right)$ otherwise. Then $EU_i\left(g'\right) > EU_i\left(f\right)$ for every $i \in C\left(g', f \mid t\right)$ and $g' \in G$. This implies that the constitution (f, F) is not ex-ante stable in G.$\square$

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