A Dual Approach to Agency Problems: Existence*

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Abstract

This paper presents a dual approach to the standard agency model. We formulate the dual problem corresponding to the principal-agent problem under the assumption that the first-order approach (FOA) is valid. This dual formulation generates a convex conjugate of a distinctive form, which transforms the agent's utility from compensation into a dual functional. The dual problem features a simple convex structure, which enables us to perform a *comprehensive* analysis for the primal problem. We derive novel and more tractable conditions for existence and uniqueness of an optimal FOA contract in terms of the functional. Furthermore, the dual approach provides us with illuminating insights into the previous nonexistence results.

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KEYWORDS: Existence, moral hazard, principal-agent models, Lagrange duality

1. Introduction

This paper concerns about the static principal-agent problem under moral hazard. The principal delegates a single task to the agent who privately chooses a productive action on behalf of the principal, but their preferred actions are not perfectly aligned. Thus, the principal designs a compensation scheme to provide a proper incentive for the agent to act in the principal's best interest. The problem of how to pay for performance can be formulated as a cost-minimization problem, together with two constraints for the agent's participation and incentive. Since its theoretical framework was developed by the seminal work by Ross (1973), the agency problem has been one of the central questions in several disciplines of economics and has been extensively studied in literature with a variety of additional constraints on payment. Unfortunately, albeit its significance,

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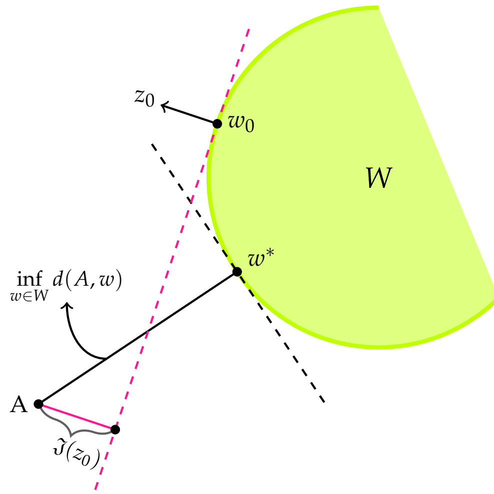


Figure 1: Geometric Illustration of the Lagrange Duality

the literature has provided no comprehensive analysis for existence of an optimal contract in this classic problem. To be specific, it is well documented that a cost minimizing contract does exist in fairly general environments when the agent's utility from compensation is bounded below at the subsistence level (e.g., Kadan, Reny and Swinkels (2017)), but on the other hand, existence may fail for different reasons when his utility is unbounded below (Mirrlees (1979) and Moroni and Swinkels (2014)). However, since the literature has adopted distinct approaches to the same problem for establishing the positive and negative results, it is rather difficult to integrate insights into the existence.

The main purpose of this paper is to develop an alternative approach to the agency problem, so that we can look into the existence issue in a full and comprehensive manner. To be precise, our approach is independent of whether the agent's utility is bounded below, or whether compensation is bounded due to limited liability, establishing existence of an optimal contract or identifying the exact cause of its failure. To this end, we employ basic tools for convex analysis (Luenberger (1969)), to formulate and analyze the Lagrange dual of the agency (or primal) problem. As in a convex optimization problem, the dual problem provides a new perspective of the primal problem at a different angle. Its comparative advantage for analysis of the agency problem is straightforward. While the primal problem involves the infinite-dimensional choice set (i.e., the *contract* space) in case of a continuous outcome distribution, which is often not easy to handle, the dual involves a finite-dimensional choice set (i.e., the set of separating *hyperplanes*). Furthermore, it turns out that in contrast with the former, the dual problem possesses nice convex properties, facilitating our analysis with the familiar tools for optimization.

The key idea of the Lagrange duality can be best understood in the context of the so-called *minimum norm problem*, which is displayed in Figure 1. For an illustration, consider the primal problem of minimizing the distance from a point (labeled A) to a convex set W. Then its Lagrange dual problem can be formulated with the distance between the point A and a supporting hyperplane separating A from the set W. Denoting each objective by $d(A, w)$ and $\mathfrak{J}(z_0)$ with the normal vector z_0 to a hyperplane, it is self-evident from Figure 1 that the shortest distance to the set W is equal to the maximum distance to a hyperplane:

$$\max \mathfrak{J}(z_0) \;=\; \inf_{w \in W} d(A, w). \tag{LD}$$

Consequently, the original minimization problem can be equivalently put as the maximization problem over hyperplanes, provided that the Lagrange duality (LD) holds. The close relationship between these two problems can be used to develop a simple criterion for existence. In fact, strong duality (LD) guarantees at least existence of a solution z_0^* to the dual problem, which in turn guarantees existence of a solution w^* to the primal problem as long as w^* is an element of W.

We apply this fundamental result to the principal-agent problem of moral hazard and derive sufficient conditions for existence or nonexistence of an optimal contract. For this purpose, we convert the principal's problem for a cost-minimizing contract into the corresponding dual problem, under the assumption that the agent's incentive constraints can be replaced by a local condition.[1] Whenever this local approach is justified, the dual is then an optimization problem over a pair of Lagrange multipliers for the participation and incentive constraints. As in a convex optimization problem, this conversion process gives rise to a convex conjugate which transforms the agent's utility from compensation into a function of the multipliers. However, the conjugate in the agency model takes a distinctive form, which we shall refer to as the *dual convex functional*. The functional is used not only to characterize the objective function of the dual problem, but to create a link to the primal problem in the sense that the functional is uniquely determined by the agent's utility function. We then establish the Lagrange duality in the agency model under regularity conditions for non-degeneracy, justifying our approach to the dual problem.

With the established duality, we next derive novel conditions for existence and its failure in terms of the dual functional instead of the agent's utility. This approach has two merits relative to the previous approaches of the primal problem. First, as is highlighted above, the dual problem features a simple framework with a two-dimensional control vector, in which we can derive conditions for existence or nonexistence in a unified and more systematic way. In particular, our conditions on the dual functional accommodate other conditions essential for existence, such as non-degeneracy of the primal problem, thereby helping us easily check on existence. Second, and more importantly, the dual approach lends better insights into the previous examples of nonexistence as well. In addition to the noted example by Mirrlees (1979), the paper by Moroni and Swinkels (2014) demonstrates that when the agent's utility is unbounded below, the primal problem is possibly degenerate and thus an optimal contract fails to exist for a different reason. Since the control vector of the dual problem indicates base pay and pay-performance sensitivity,

[1] This first-order approach (FOA) does restrict the agent's possible deviations from the recommended action and is typically valid under restrictive conditions (e.g., Rogerson (1985), Jewitt (1988) and Conlon (2009) among others). In response, recent studies such as Renner and Schmedders (2015) and Ke and Ryan (2018) develop alternative methods of characterizing an optimal contract without relying on FOA, but nonetheless the approach is still widely adopted by practitioners.

our analysis does not only provide a clear intuition behind the previous nonexistence results but helps identify a condition for possible degeneracy in other environments.

This paper contributes to the literature on existence of an optimal incentive-compatible contract in the principal-agent model with moral hazard. While the previous literature (e.g., Holmström (1979), Page (1987), Kadan *et al.* (2017) among others) seeks to establish the existence through analysis of the principal's primal problem, we examine this significant issue from the dual perspective. As our dual analysis is based on validity of the first-order approach in the primal problem, our results on existence in Section 4.1 (where we address the models with the agent's utility bounded below) are rather restrictive compared to the general existence result of Kadan *et al.* (2017). However, our aim in the current paper is not to establish existence in a more general setting but to develop a unified approach to the issue in the most rudimentary agency model. There are two reasons for this. First, the approach yields a simple condition for existence in terms of the dual element. In particular, since our condition is also sufficient for duality (LD) to hold, we can dispense with regularity conditions such as non-degeneracy. Second, and more importantly, the same approach is applicable even when the agent's utility is unbounded below, to derive conditions even for nonexistence of an optimal contract. Therefore, the unified approach helps gain better insights into the previous nonexistence results as well.

Our dual approach is inspired by Jewitt (2007) and Jewitt, Kadan and Swindkels (2008), both of which formulated the dual problem from the standard convex conjugate, also known as the Lagrange-Fenchel transform, under the assumption that the given agency problem is non-degenerate so duality holds. However, as we can deduce from Moroni and Swinkels (2014), the regularity condition is not always satisfied in an agency model, for example, in case of bounded payment due to limited liability or bounded likelihood ratios. Our work based on a distinct type of conjugate generalizes their approaches in that we do not assume non-degeneracy of the primal problem; instead, we identify tight conditions on the dual convex functional which ensure existence of a feasible contract, thereby establishing the close relationship between the primal and dual problems.

The rest of the paper is organized as follows. Section 2 presents the principal's primal problem of minimizing a cost for implementation of her desired action. Section 3 introduces the dual element of the agent's utility function, which is used to formulate the dual problem, and then establishes duality between the primal and dual problems under relaxed regularity conditions. Applying the duality principles, Section 4 derives conditions on the dual convex functional under which an optimal contract exists in a unique form or fails to exist. Section 5 concludes.

2. The Agency Problem

We consider the standard agency problem in which a risk-neutral principal (she) delegates a single task to a risk-averse agent (he), taking advantage of the agent's expertise. The principal offers the agent a contract which specifies a compensation schedule on the basis of verifiable score $s \in \mathcal{S} \subset$

$\Re$, where the scalar s indicates the given performance measure.[2] Accepting the contract, the agent supplies an input $a \in A = [0, \bar{a}]$ for the delegated task at a personal cost of $\psi(a)$, and this move is unobserved by the principal.[3] Declining the contract, he obtains a reservation payoff of $\underline{V}$. The agent's input a determines the distribution of his score, $F(s|a) \equiv \Pr(S \leq s|a)$.[4] We make the standard assumptions that the support of S is independent of a and F admits a density function f with respect to Lebesgue measure.

After the contract is executed, the agent obtains a payoff of $u(w) - \psi(a)$. The first term $u : [\underline{w}, \infty) \to \Re$ represents the agent's utility from compensation, which is assumed increasing and strictly concave. Our subsequent analysis of the model is comprehensive, so that we allow the function u to be bounded or unbounded at the minimum wage $\underline{w}$, and allow u to be increasing with or without bound. The second term ψ indicates the monetary cost of supplying input a to the agent, which is assumed nondecreasing. The agent's payoff is additive separable, implying that the agent's preference over uncertainty is independent of his input. Given a contract w, the agent's expected payoff can be written as $V(w, a) \equiv \mathbb{E}[u(w(S))|a] - \psi(a)$.

To induce input a from the agent at the least cost, the principal must offer a contract that solves the following optimization problem:

$$\min_{w \geq \underline{w}} \mathbb{E}[w(S)|a] \tag{P}$$

subject to the participation and incentive constraints,

$$V(w, a) \geq \underline{V} \tag{PC}$$

$$V(w, a) \geq V(w, a') \quad \forall\, a' \in A. \tag{IC}$$

One important issue which has been extensively studied in the literature is how to sort out relevant ones among the set of incentive-compatible (IC) constraints. In what follows, we assume that the first-order approach (FOA) is valid, so that relevant is only the local downward IC constraint.[5] Under this approach, the set of IC constraints can be replaced by the following relaxed condition (Rogerson (1985)):

$$\frac{\partial}{\partial a} V(w, a) \equiv V_a(w, a) = \mathbb{E}[u(w(S)) L^a(S)|a] - \psi'(a) \geq 0, \tag{L-IC}$$

where $L^a(s) = \frac{\partial}{\partial a} \log f(s|a)$ denotes the likelihood ratio at the desired action a. In case of a discrete action space $\{a_1, a_2, \cdots, a_K\}$, the relevant local constraint for implementation of a_k becomes

[2] We present the model with a unidimensional signal for simple exposition, but the results are readily extended to multidimensional signals as long as the first-order approach is valid.

[3] Our results continue to hold even if the agent's action space A is a discrete set.

[4] Throughout the paper a capital letter S is used for a random variable, and a small letter s is used for its realization.

[5] Existing literature has developed a various set of conditions on either the agent's utility function u or the distribution F (or both), under which this local approach can be justified. One of the most common conditions is the CDFC (convexity of the distribution function) property in case of univariate performance measures, which requires $F_{aa}(\cdot|a) \leq 0$. In case of multivariate measures, Conlon (2009) has shown that if F satisfies the monotone likelihood ratio property (MLRP) and the concave increasing set probability (CISP), then the FOA is valid. For other conditions justifying the FOA, refer to Jung and Kim (2015) and Kirkegaard (2017).

$V(w, a_k) \geq V(w, a_{k-1})$ under FOA. Put it in words, the approach postulates that at the optimal contract, there is only a local downward deviation from the intended action, and therefore the constraint (L-IC) precludes the agent's all possible deviations.

In order to clarify the agency problems of interest, we define

$$\Omega \equiv \Big\{ \langle u, \psi, F \rangle \,\big|\, u : [\underline{w}, \infty) \to \Re \text{ increasing and strictly concave,}$$
$$\psi : A \to \Re \text{ nondecreasing, FOA is valid under } F \Big\}.$$

Each element of the set Ω collects the key components relevant to the agent's payoff, identifying an agency problem together with the distribution of outcomes F. For each $\langle u, \psi, F \rangle$, the main problem for the principal is then to minimize the expected pay necessary for implementing the desired action, given the PC and L-IC constraints. Hereafter, we call this constrained minimization problem (P) the *primal problem*, and we say that a contract $w(s) \geq \underline{w}$ is *feasible* if it satisfies the two constraints. Since the agent's utility function is concave, it is immediate from Jensen's inequality that the set of feasible contracts is convex. Hence every primal problem specified by an element of Ω can be put as a convex optimization problem.

If the primal problem admits a solution w^*, we denote by $C^*(a) = \mathbb{E}[w^*(S)|a]$ the primal value. The value $C^*(a)$ represents the minimal expected pay necessary for implementation of the desired action a under moral hazard. For a benchmark, let $C^F(a)$ denote the minimal cost under perfect information. When the agent's behavior is observable, the cost-minimizing contract features fixed pay w^F irrespective of his performance score s. We assume that at least such a first-best contract exists for every problem in Ω, and thus the participation constraint (PC) alone does not pose an agency problem. On the other hand, under moral hazard, the principal inevitably has to offer a contract contingent upon the uncertain outcomes for incentive provision. In exchange for shifting payoff risks to the agent, the principal gives up a risk premium which gives rise to an agency cost as much as $C^*(a) - C^F(a) \geq 0$.

3. Duality in Agency Models

In this section, we formulate the Lagrange dual of the agency problem (P) and establish their equivalence, the so-called Lagrange *duality*.

3.1. Weak Duality

Given an agency problem $\langle u, \psi, F \rangle \in \Omega$, define the primal value function as

$$\mathfrak{C}(a) \equiv \inf_{w \geq \underline{w}} \{\mathbb{E}[w(S)|a] : w \text{ is a feasible contract}\}.$$

Notice that in contrast to $C^*(a)$, the value function $\mathfrak{C}(a)$ is defined as the *infimum* of expected pay and thus it is always well-defined irrespective of existence of an optimal contract. To be specific, we have $C^*(a) = \mathfrak{C}(a) < \infty$ whenever an optimal contract exists, whereas we have $\mathfrak{C}(a) = \infty$ if there exist no contracts feasible. In the latter case, we say that the primal problem is degenerate or infeasible.

We next construct the associated Lagrangian $\mathfrak{L}$ with the primal problem and combine the terms dependent on w, to write it as follows:

$$\begin{aligned}\mathfrak{L}(w;\mathbf{z}) \;&\equiv\; \mathbb{E}[w(S)\,|\,a] + \mathbf{z}\cdot\left(\underline{V} - V(w,a),\, -V_a(w,a)\right)\\ &=\; \mathbb{E}\big[w(S) - u(w(S))\,(z_1 + z_2 L^a(S))\,|\,a\big] + z_1(\psi(a)+\underline{V}) + z_2\psi'(a),\end{aligned} \tag{1}$$

where $\mathbf{z} = (z_1, z_2) \geq \mathbf{0}$ are the Lagrange multipliers for the constraints PC and L-IC, respectively. With the Lagrangian, we define the counterpart of $\mathfrak{C}(a)$ as

$$\mathfrak{J}(\mathbf{z}) \;\equiv\; \inf_{w \geq \underline{w}} \; \mathfrak{L}(w;\mathbf{z}), \tag{2}$$

which is referred to as the *dual value function* hereafter. Then it directly follows by definition that $\mathfrak{J}(\mathbf{z}) \leq \mathfrak{L}(w;\mathbf{z}) \leq \mathbb{E}[w(\mathbf{X})|a]$ for every feasible contract w and for all nonnegative multipliers $\mathbf{z}$. This leads us to the next inequality, the so-called *weak duality*:

$$\max_{\mathbf{z}\geq\mathbf{0}} \mathfrak{J}(\mathbf{z}) \;\leq\; \mathfrak{C}(a). \tag{wD}$$

To put it in words, the dual value function $\mathfrak{J}(\mathbf{z})$ provides a lower bound on the primal value and weak duality holds in general. In particular, the inequality (wD) is satisfied even when the dual value function is increasing without bound, so that $\sup_{\mathbf{z}\geq\mathbf{0}} \mathfrak{J}(\mathbf{z}) = \infty$. Then $\mathfrak{C}(a) = \infty$ is immediate from weak duality, implying that the primal problem is infeasible.

3.2. The Dual Convex Functional

The dual value function $\mathfrak{J}$ defined in (2) involves an optimization problem over a functional space, determining a compensation scheme for all possible outcomes. This infinite-dimensional problem is usually difficult to deal with, and moreover, the expectation in (1) is not even defined for a certain class of contracts. In order to circumvent this difficulty, we adopt the standard method of taking the pointwise infimum in (2) to characterize the function $\mathfrak{J}$. This process simplifies the problem into a uni-dimensional one of assigning a payment to each realized score s, or equivalently, to each value of the informational variable $q_{\mathbf{z}}(S) \equiv z_1 + z_2 L^a(S)$ given the multipliers.

The simplified problem can be put as a family of maximization problems with parameter q:

$$\phi(q) \;\equiv\; \sup_{w\geq\underline{w}} \left\{u(w)q - w\right\}. \tag{T}$$

Consequently, the pointwise process gives rise to the value function ϕ for each value of q, and the process can be viewed as a transformation $T : u \mapsto \phi$ from the primitive u to a function ϕ of the single variable q. With slight abuse of terminology we refer to ϕ as the *dual convex functional*, and denote by $w^*(q) \geq \underline{w}$ a solution to the maximization problem above.[6] The transform T is at the heart of our subsequent analysis on existence of an optimal contract, as it turns out.

We next enumerate basic properties of the functional ϕ. In light of definition, it comes as no surprise that the properties rest on the agent's utility function.

Lemma 1. *The dual convex functional ϕ possesses the following properties:*

(i) *$\phi(q)$ is finite-valued for $q \leq 0$ if $u(\underline{w}) > -\infty$.*

(ii) *$\phi(q)$ is finite-valued on $q \geq 0$ if $\lim_{w \uparrow \infty} u'(w) = 0$.*

(iii) *ϕ is convex in the region where it is finite.*

PROOF OF LEMMA 1: See Appendix A.2. □

We next categorize the agent's utility function into three types and for each type, we illustrate the distinctive features of the dual convex functional via examples.

Example 1. *Suppose that the agent's utility function is given by $u(w) = w^\kappa$ defined on $[0, \infty)$, where $\kappa \in (0, 1)$ is a constant. Then its dual convex functional ϕ takes a form of*

$$\phi(q) = \begin{cases} 0 & \text{for } q < 0 \\ \left(\frac{1}{\kappa} - 1\right)(\kappa q)^{1/(1-\kappa)} & \text{for } q \geq 0. \end{cases}$$

The utility function of the form w^κ on $[0, \infty)$ is bounded below and continuously increasing with no linear asymptotes. As a result, it follows from Lemma 1 that the dual convex function is finite everywhere, hence globally convex. In particular, as ϕ is strictly convex in the region $[0, \infty)$, the optimal solution $w^*(q)$ characterizing ϕ is uniquely determined for each $q \geq 0$.

Example 2 (Bounded Utility/Payments)**.**

(a) Let $u(w) = 1 - e^{-\kappa w}$ on $[0, \infty)$, where $\kappa > 0$ is a constant. Its dual convex functional ϕ takes a value of zero for $q \leq \kappa^{-1}$ and $q - \kappa^{-1}[1 + \ln \kappa q]$ for $q > \kappa^{-1}$.

(b) Let $u(w) = 2\sqrt{w}$ on $[0, \overline{w}]$. Then the functional $\phi(q)$ takes a value of zero in $(-\infty, 0)$, q^2 in $[0, \overline{w}^{1/2}]$, and $2\overline{w}^{1/2}q - \overline{w}$ in $(\overline{w}^{1/2}, \infty)$.

The second example delineates a contracting environment where the agent's utility is not much responsive to a large payment or the payment itself is bounded due to limited liability (Jewitt *et al.* (2008)). The utility functions in this environment are uniformly bounded, and as a result, the dual convex function ϕ takes a finite value everywhere as in the previous example. However, bounded utility gives rise to one notable difference. Since $\phi'(q) = u(w^*(q))$ holds by the envelope

[6] The dual transform T differs from the Legendre-Fenchel transform (or the *convex conjugate*), which takes a form of $u^*(q) = \sup_w \{qw - u(w)\}$ in our framework. See Jewitt (2007) who applies this standard transform to develop a dual approach to the agency problem in the same spirit as ours.

theorem, the functional ϕ has its derivative uniformly bounded. It turns out that whether ϕ' is bounded is a decisive factor for existence of an optimal contract. To be specific, we demonstrate in the next section that an optimal contract does not necessarily exist if there is a constant M such that $|\phi'(q)| < M$ for every q.

In contrast with the examples above, the dual convex functional can take infinity as a value on its domain. Lemma 1 tells us that this is the case if u is unbounded below or increasing along an oblique asymptote.

Example 3 (Unbounded/Asymptotically Linear Utility)**.**

(a) Let $u(w) = -\frac{1}{w}$ on $[0,\infty)$. Then $\phi(q) = -2\sqrt{q}$ for every $q \geq 0$, whereas $\phi(q) = \infty$ for every $q < 0$.

(b) Let $u(w) = 2\sqrt{w}$ on $[0,1]$ and $w+1$ on $[1,\infty)$. Then ϕ takes a value of 0 in $(-\infty,0)$, q^2 in $[0,1]$, and ∞ in $(1,\infty)$.

As is shown in this example, the dual convex functional takes ∞ as a value on its domain if the agent cannot fulfill his subsistence needs at the current minimum wage, or if the agent tends to be neutral for a compensation risk when base pay is high enough.[7] In particular, if the likelihood ratio L^a is unbounded, the variable $q = z_1 + z_2 L^a(S)$ with $z_2 > 0$ would traverse the region where $\phi = \infty$ with positive probability. It can be shown that in this case, an optimal contract does not exist for the same reason as in Mirrlees (1979). Furthermore, even if the ratio is bounded, existence may fail for a different reason as it has been observed by Moroni and Swinkels (2014). In the next section we examine the existence problem by adopting an alternative approach, to develop further insights into the previous results.

3.3. The Duality Theorem

Substituting the dual convex functional ϕ into the Lagrangian $\mathfrak{L}$, we obtain a closed form of the dual value function:

$$\mathfrak{J}(\mathbf{z}) \;=\; -\mathbb{E}\left[\phi\left(z_1 + z_2 L^a(S)\right) \mid a\right] + z_1(\psi(a) + \underline{V}) + z_2\psi'(a) \;=\; \mathfrak{L}(w_{\mathbf{z}};\mathbf{z}), \tag{3}$$

where $w_{\mathbf{z}}$ is the contract derived from the functional ϕ, thereby satisfying $w_{\mathbf{z}}(s) = w^*(q_{\mathbf{z}}(s))$ for every realized score s. With the function $\mathfrak{J}$, we can state the Lagrange *dual problem* as:

$$\max_{\mathbf{z}\geq\mathbf{0}} \; \mathfrak{J}(\mathbf{z}). \tag{D}$$

We denote by $\mathbf{z}^* = (z_1^*, z_2^*)$ a solution to the dual problem, if it exists. Since $\mathfrak{J}(\mathbf{z}) \leq \mathfrak{C}(a)$ follows by weak duality, the dual value $\mathfrak{J}(\mathbf{z}^*)$ serves as a lower bound for the primal value. In general, there is a gap between these two values, which is referred to as the duality gap. In this subsection, we

[7] Simple algebra reveals that if the utility function has an oblique asymptote, i.e., $\lim_{w\to\infty} u'(w) = \frac{1}{\kappa}$ for some $\kappa > 0$, then $\phi(q) = \infty$ for every $q > \kappa$.

present a pair of conditions under which the gap disappears; in other words, *strong duality* holds within the set Ω of agency problems.

The most well-known condition for strong duality is presumably Slater's condition, which requires in our framework existence of a contract $w(\mathbf{x}) \geq \underline{w}$ such that the two constraints both have slack:

$$-V(w,a) + \underline{V} \ < \ 0 \quad \text{and} \quad -V_a(w,a) \ < \ 0. \tag{4}$$

Put differently, Slater's condition calls for existence of a strictly feasible contract. It is well-known that so long as the primal problem is feasible, the condition (4) is sufficient for strong duality to hold for a convex optimization problem (Theorem 1 in Chapter 8.6 of Luenberger (1969)). For our purpose, however, we generalize the standard condition a bit further to establish the duality principle:

Theorem 1 (Lagrange Duality in Agency Models)**.** *Suppose that an agency problem* $\langle u, \psi, F \rangle \in \Omega$ *fulfills the next two conditions:*

(i) $\mathfrak{C}(a) < \infty$, *and*

(ii) *for each* $\mathbf{z} > \mathbf{0}$, *there exists a* $w_{\mathbf{z}} \geq \underline{w}$ *such that* $\mathbf{z} \cdot (\underline{V} - V(w_{\mathbf{z}}, a), \ -V_a(w_{\mathbf{z}}, a)) < 0$.

Then there exists a solution $\mathbf{z}^* \geq \mathbf{0}$ *to the dual problem* (D), *and furthermore, strong duality holds:*

$$\mathfrak{C}(a) \ = \ \max_{\mathbf{z} \geq \mathbf{0}} \ \mathfrak{J}(\mathbf{z}). \tag{SD}$$

PROOF OF THEOREM 1: See Appendix A.1. □

Theorem 1 establishes strong duality in the agency model under the two conditions. Condition (i) is a fairly mild condition requiring that the primal problem be feasible, whereas (ii) is a rather restrictive but essential condition for duality which deserves comment. In line with other non-degeneracy conditions, the condition (ii) ensures existence of a nontrivial separating hyperplane. However, the condition generalizes Slater's condition (4) in that it admits a possibility that one of the two given constraints is violated. For an illustration, suppose that there exists a contract w_1 for which only the incentive constraint is met, i.e., $V_a(w_1, a) \geq 0$ but $V(w_1, a) < \underline{V}$, so that existence of w_1 does not directly imply (4). Nonetheless, condition (ii) does hold, provided that the incentive constraint has slack large enough to nullify the participation constraint. In the next section, we make use of this relaxed condition to establish strong duality for the agency problem, even when the dual value function $\mathfrak{J}$ - the objective of the dual problem - is not well-behaved, thereby justifying our approach to the dual problem in a large class of the agency problems.

Theorem 1 tells us that the two different problems yield the same value, and thus for each agency problem in Ω, we can identify the exact agency cost from the dual problem (D). Compared to the primal problem (P), the dual features a rather simple structure with a pair of real-valued choice variables $\mathbf{z} = (z_1, z_2)$, which indicate in the agency problem base pay and pay-performance sensitivity, respectively. Furthermore, strong duality serves as a toolkit for our analysis of existence of an optimal contract in the next section, as is shown in the next result:

Corollary 2. *Suppose that strong duality (SD) holds for an agency problem $\langle u, \psi, F\rangle \in \Omega$, and let w^* denote the contract derived from the dual problem:*

$$\max_{\mathbf{z}\geq\mathbf{0}} \mathfrak{J}(\mathbf{z}) \;=\; \mathfrak{J}(\mathbf{z}^*) \;=\; \mathfrak{L}(w^*;\mathbf{z}^*).$$

If the contract w^* is feasible, then w^* constitutes an optimal contract.

PROOF OF COROLLARY 2: Denote by $\mathbf{z}^*$ a solution to the dual problem. Observe that under the given conditions, we have

$$\mathbb{E}[w^*(S)|a] \;\leq\; \mathfrak{C}(a) \;=\; \mathfrak{J}(\mathbf{z}^*) \;=\; \mathfrak{L}(w^*;\mathbf{z}^*), \tag{5}$$

where the inequality is due to the assumption that w^* is a feasible contract. In addition, it follows from $\mathbf{z}^* \geq \mathbf{0}$ that

$$\mathfrak{L}(w^*;\mathbf{z}^*) \;=\; \mathbb{E}[w^*(S)|a] + \mathbf{z}^* \cdot (\underline{V} - V(w^*,a),\; -V_a(w^*,a)) \;\leq\; \mathbb{E}[w^*(S)|a]. \tag{6}$$

Then putting (5) and (6) together results in $\mathfrak{C}(a) = \mathbb{E}[w^*(S)|a]$. $\square$

Before turning to the next section, it is worthwhile to see that strong duality holds at least under perfect information. Absent the incentive constraint, the first-best dual problem can be formulated with a single multiplier z as $\max_{z\geq 0} \mathfrak{J}^F(z)$, where $\mathfrak{J}^F(z) = -\phi(z) + z[\psi(a) + \underline{V}]$. Hence $\mathfrak{J}^F(z) = \mathfrak{J}(z,0)$ is immediate from (3). Since we assume existence of a first-best contract for every problem in Ω, it follows by strong duality that $C^F(a) = \mathfrak{J}(z^F, 0)$ for some $z^F \geq 0$.

4. Existence of an Optimal Contract

In this section, we apply the duality principles established in Section 3 and derive novel conditions for existence of an optimal contract in the standard agency model. The dual problem provides a different angle for analysis, and moreover, its simple structure allows us to look into the existence issue in a systematic manner. To begin with, in light of the examples in Section 3.2, we subdivide our analysis into two cases depending on whether the dual convex functional ϕ is finite-valued.

4.1. Finite-valued Dual Convex Functional

We first deal with the agency model whose primitives give rise to a finite-valued dual convex functional ϕ on its domain. Recall that this is the case if and only if the agent's utility at the minimum wage $\underline{w}$ is bounded below and his marginal utility from compensation w converges to zero as w grows large. We derive sufficient conditions for the existence in terms of the functional ϕ using the structure of the dual problem.

For this purpose, by an appropriate translation, we assume without loss of generality that both $\underline{w}$ and $u(\underline{w})$ are zero. This normalization simplifies the functional ϕ, which takes a value of zero on $(-\infty, 0]$ and is increasing and convex on $[0, \infty)$. As a result, it follows from Lemma 1 that ϕ is globally convex on its domain, and therefore, the dual value function $\mathfrak{J}$ put in (3) is twice differentiable almost everywhere.

Taking the partial derivative of $\mathfrak{J}$ with respect to z_2, we obtain

$$\mathfrak{J}_2(\mathbf{z}) \;=\; -\mathbb{E}\left[\phi'(q_{\mathbf{z}}(S))L^a(S)|a\right] + \psi'(a) \;=\; -V_a(w_{\mathbf{z}}, a), \tag{7}$$

where $q_{\mathbf{z}}(S) = z_1 + z_2 L^a(S)$ and the last expression is due to the envelope theorem. Hence the partial derivative $\mathfrak{J}_2$ represents the (negative) marginal net gain from input to the agent given the contract $w_{\mathbf{z}}$, and thus $\mathfrak{J}_2 \leq 0$ indicates that $w_{\mathbf{z}}$ fulfills the incentive constraint (L-IC).

To better understand how the function $\mathfrak{J}$ responds to a change in z_2, observe that when $z_2 = 0$, the informational variable $q_{\mathbf{z}}$ boils down to a constant z_1, so that the corresponding contract $w_{\mathbf{z}}$ represents fixed pay. We know that such a contract violates the incentive constraint. As a matter of fact, substituting $z_2 = 0$ into (7) yields $\mathfrak{J}_2(z_1, 0) = \psi'(a) > 0$ for all z_1, because the likelihood ratio $L^a(S)$ has mean zero. This reveals that $z_2 = 0$ can never be a solution to the dual problem. In addition, since ϕ is globally convex and strictly convex on a set of positive measure, we have the second partial derivative negative:

$$\mathfrak{J}_{22}(\mathbf{z}) \;=\; -\mathbb{E}\left[\phi''(q_{\mathbf{z}}(S))\,(L^a(S))^2\,|a\right] \;<\; 0.$$

Consequently, given the contract $w_{\mathbf{z}}$, the agent's marginal gain from input is increasing in z_2. The idea behind this negative sign is simple. An increase of z_2 in the contract $w_{\mathbf{z}}$ is tantamount to the compensation scheme being more sensitive to performance scores, strengthening the agent's incentives. Since $\mathfrak{J}_2$ turns out to be monotone decreasing in z_2 and $\mathfrak{J}_2(z_1, 0) > 0$, we can deduce that for each z_1, there exists a unique $z_2^* > 0$ satisfying $\mathfrak{J}_2(z_1, z_2^*) = 0$, provided that $\lim_{z_2 \uparrow \infty} \mathfrak{J}_2 < 0$.

Similarly, we compute the first- and second-order partial derivatives with respect to z_1 to obtain

$$\mathfrak{J}_1 \;=\; -V(w_{\mathbf{z}}, a) + \underline{V} \qquad \text{and} \qquad \mathfrak{J}_{11} \;=\; -\mathbb{E}\left[\phi''(q_{\mathbf{z}}(S))\,|\,a\right] \;<\; 0.$$

The partial derivative $\mathfrak{J}_1$ yields the agent's (negative) net expected payoff from the contractual relationship enforced by $w_{\mathbf{z}}$. The negative sign of $\mathfrak{J}_{11}$ is again due to convexity of ϕ, implying that the payoff is increasing in the level of pay.

Our discussion so far leads us to the next lemma:

Lemma 2. *Suppose that an agency problem $\langle u, \psi, F\rangle \in \Omega$ gives rise to a finite-valued dual convex functional ϕ. Then the dual value function $\mathfrak{J}$ is strictly concave, so that a solution to the dual problem* $\max \mathfrak{J}(\mathbf{z})$*, if it exists, is uniquely determined at the stationary point of $\mathfrak{J}$.*

PROOF OF LEMMA 2: By the Cauchy-Schwarz inequality, we have

$$(\mathfrak{J}_{12})^2 = \left(\mathbb{E}\left[\phi''(q_{\mathbf{z}}(S))L^a(S)|a\right]\right)^2 \;<\; \mathbb{E}\left[\phi''(q_{\mathbf{z}}(S))|a\right]\mathbb{E}\left[\phi''(q_{\mathbf{z}}(S))\left(L^a(S)\right)^2|a\right] = \mathfrak{J}_{11}\mathfrak{J}_{22}.$$

The strict inequality above and $\mathfrak{J}_{22} < 0$ reveal that the Hessian matrix of $\mathfrak{J}$ is negative definite. As a result, $\mathfrak{J}$ has at most one stationary point where $\mathfrak{J}$ attains a global maximum. □

With the properties of $\mathfrak{J}$ established, we are now ready to state our first existence result:

Proposition 3. *Suppose that the dual of an agency problem $\langle u, \psi, F\rangle \in \Omega$ results in a finite-valued dual convex functional ϕ with ϕ'' weakly monotone (i.e., either nondecreasing or nonincreasing). Then an optimal contract exists in a unique form if*

$$\lim_{z_2\uparrow\infty} \mathfrak{J}_2(z_1, z_2) \;<\; 0 \quad \textit{for every } z_1 \geq 0. \tag{8}$$

PROOF OF PROPOSITION 3: See Appendix A.3. □

The proof of Proposition 3 is composed of two parts. We first show that a weakly monotone ϕ'' and the condition (8) are sufficient for the dual value function $\mathfrak{J}$ to have a stationary point $\mathbf{z}^*$. Hence it follows by Lemma 2 that the dual problem has a unique solution at $\mathbf{z}^*$. Based on this property of $\mathfrak{J}$, we then employ strong duality to establish existence of an optimal contract. Since the dual approach maps a pair of the Lagrange multipliers onto a contract through the transform T, we obtain the corresponding contract $w_{\mathbf{z}^*}$ to the unique stationary point. On top of that, at this contract the two constraints are binding because $\nabla\mathfrak{J}(\mathbf{z}^*) = \mathbf{0}$. Optimality of $w_{\mathbf{z}^*}$ is then an immediate consequence of Corollary 2.

Proposition 3 presents the two conditions for existence in Ω where the primitives yield a finite-valued dual convex functional ϕ. Compared to the condition that ϕ'' be weakly monotone, rather strong is the second condition (8) which requires that the principal is able to offer adequate incentives through a contract of form $w_{\mathbf{z}}$ by adjusting pay-performance sensitivity z_2.[8] Moreover, the condition is difficult to verify in that the limiting behavior of $\mathfrak{J}$ naturally depends on the distribution F as well as the agent's utility function u. A follow-up question is therefore for which u we have $\mathfrak{J}_2$ falling below zero. Our next result shows that (8) holds if ϕ has an unbounded derivative; namely, if the agent's utility function is unbounded above.

Corollary 4. *Suppose that the dual of an agency problem $\langle u, \psi, F\rangle \in \Omega$ gives rise to a finite-valued dual convex functional ϕ satisfying two properties:* (i) $\lim_{q\uparrow\infty}\phi'(q) = \infty$ *and* (ii) ϕ'' *is weakly monotone. Then an optimal contract exists in a unique form.*

PROOF OF COROLLARY 4: By Proposition 3, it suffices to show that $\lim_{z_2\to\infty}\mathfrak{J}_2(\mathbf{z}) < 0$. To this end, observe that for a given $z_1 \geq 0$, as z_2 grows large, the sign of the variable $q_{\mathbf{z}}(s) = z_1 + z_2 L^a(s)$ is governed by the sign of the likelihood ratio $L^a(s)$. To be precise, we

[8]Refer to Example 4 and Proposition 6 for a discussion of the condition that ϕ'' be weakly monotone.

have $|\Pr(q_{\mathbf{z}}(S) \geq 0|a) - \Pr(L^a(S) \geq 0|a)| \to 0$. Furthermore, $\phi'(z_1 + z_2 L^a(S))$ is increasing in z_2 whenever $L^a \geq 0$, so it follows by the dominated convergence theorem that if ϕ' is unbounded above, the expectation in (7) grows arbitrarily large as $z_2 \to \infty$. Therefore, $\mathfrak{J}_2$ falls below zero. □

Corollary 4 provides us with a pair of novel and simple conditions for existence and uniqueness in terms of the dual element ϕ. It is worthwhile to see that Corollary 4 is independent of the distribution F and, in particular, the result dispenses with other conditions necessary for non-degeneracy of the primal problem. The role of the condition $\phi' = u \to \infty$ is straightforward. If u is unbounded above, then the incentive-compatibility constraint can be fulfilled by increasing z_2 within a class of contracts $w_{\mathbf{z}}$, put it in words, by offering a contract more responsive to the outcome. In fact, the unbounded ϕ' is essential for the existence. In case of bounded derivatives as in Example 2, it is possible that any contracts of the form $w_{\mathbf{z}}$ do not provide a proper incentive for the intended action; in other words, $\mathfrak{J}_2$ remains positive for every $\mathbf{z}$, and thus existence inevitably fails.

Corollary 5. *Suppose that an agency problem $\langle u, \psi, F\rangle \in \Omega$ results in the finite-valued dual convex functional ϕ with $|\phi'| < M$ for some constant M. Then an optimal contract does not exist if*

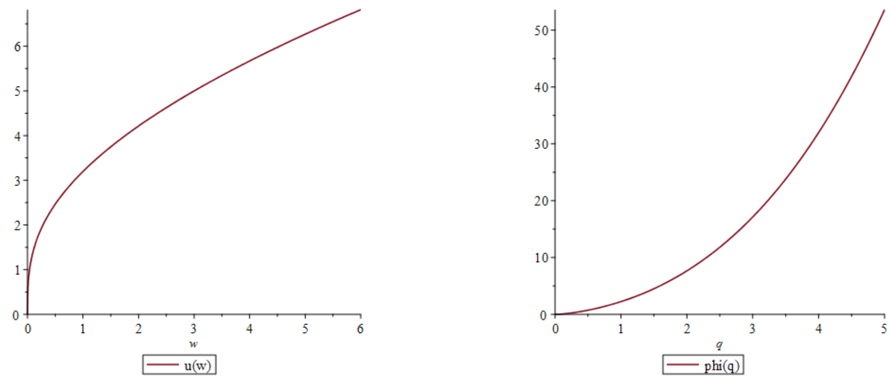
$$\sqrt{\mathbb{E}[L^a(S)^2|a]} \;\leq\; \frac{\psi'(a)}{M}. \tag{9}$$

PROOF OF COROLLARY 5: Note that for every $z_2 > 0$, the expectation in the expression (7) of $\mathfrak{J}_2$ is bounded above by

$$\mathbb{E}\left[\phi'(q_{\mathbf{z}}(S))L^a(S)|a\right] \;\leq\; \sqrt{\mathbb{E}\left[\phi'(q_{\mathbf{z}}(S))^2|a\right]}\sqrt{\mathbb{E}\left[L^a(S)^2|a\right]} \;<\; M\sqrt{\mathbb{E}\left[L^a(S)^2|a\right]},$$

where the first inequality is due to the Cauchy-Schwarz inequality. Consequently, if (9) holds, we have $\mathfrak{J}_2 > 0$ for every $\mathbf{z}$. This implies that the dual value function $\mathfrak{J}$ is continuously increasing in z_2 without upper bounds, and thus $\mathfrak{C}(a) = \infty$ follows by weak duality. Therefore, the primal problem is infeasible and an optimal contract does not exist. □

Corollary 5 tells us that if ϕ has a bounded derivative, existence hinges upon how much diffuse the likelihood ratio L^a is. To gain insights, recall that the random variable $L^a(S)$ contains information about the agent's hidden behavior. When the variable has small variance, it is not much sensitive to the agent's possible (local) deviations; put differently, the given performance metric S contains little information about his choice of input. Consequently, no contracts of the form $w_{\mathbf{z}}(s)$ does meet the incentive constraint, and the primal problem is therefore infeasible. For a concrete example, consider the standard agency problem in which the agent privately controls the mean of the normal distribution $S \sim N(\mu + a, \sigma^2)$ at a cost of $\psi(a)$. The likelihood ratio of this signal follows a normal distribution with mean zero and variance σ^{-2}, regardless of the intended action a. Therefore, when the agent's utility function is given by $u(w) = 1 - e^{-w}$, there exist no optimal contracts implementing a if $\psi'(a) \geq \sigma^{-1}$.



(a) $u(w) = \inf_{q>0}\{q^{-1}(\phi(q) + w)\}$ (b) $\phi(q) = \sup_{w\geq 0}\{u(w)q - w\}$

Figure 2: Duality between agent's utility and dual convex functions

So far we remain silent about the assumption that ϕ'' is weakly monotone, notwithstanding its importance for our existence results. Together with (8), monotone ϕ'' is essential to rule out the possibility that the strictly concave function $\mathfrak{J}$ has no stationary points. Compared to the other conditions, this assumption is fairly innocuous and is indeed satisfied for typical utility functions including Examples 1 and 2. However, as is demonstrated in the next example, we may have non-monotone ϕ'' for some unusual functions:

Example 4. *Consider the following utility function defined as*

$$u(w) \;=\; \inf_{q>0}\left\{2\sqrt{q} + \frac{1}{4}q^2 + \frac{w}{q}\right\} \;=\; \int_0^w \left[-1+\sqrt{1+2s}\right]^{-2/3} ds \quad \text{for } w \in [0,\infty),$$

which is continuously increasing, $\lim_{w\to\infty} u'(w) = 0$, strictly concave, and bounded below as is displayed in Figure 2-(a). The associated dual convex functional takes a form of

$$\phi(q) \;=\; \begin{cases} 2q^{3/2} + \frac{1}{4}q^3 & \text{for} \quad q \geq 0, \\ 0 & \text{for} \quad q < 0. \end{cases}$$

In fact, the function u is obtained by taking the inverse transform of T on the functional ϕ, $u(w) = \inf_{q>0}\{q^{-1}(\phi(q)+w)\}$, which follows from the definition of T, á la Fenchel's inequality.[9] *Although globally convex, the functional ϕ has a non-monotone second-order derivative.*

When monotonicity of ϕ'' fails as in Example 4, neither Proposition 3 nor Corollary 4 are unfortunately applicable. Our last result in this subsection supplements the preceding results, establishing existence of an optimal contract even when ϕ'' behaves non-monotonically.

Proposition 6. *Suppose that the dual of an agency problem $\langle u, \psi, F\rangle \in \Omega$ gives rise to a finite-valued dual convex functional ϕ which is strongly convex on $[0,\infty)$: there exists a constant $m > 0$ such that $\phi''(q) > m$*

[9] In the companion paper Chi and Choi (2021), we go into detail about the inverse transform and establish duality between u and ϕ. The duality implies that an agency problem can be equivalently stated in terms of u or ϕ.

for every $q \in [0, \infty)$. Then an optimal contract exists in a unique form.

PROOF OF PROPOSITION 6: See Appendix A.4. $\square$

The basic idea behind Proposition 6 stems from the concept of strong convexity. It is well-known that if the objective function defined on a closed set is strongly convex (strongly concave), its minimization (maximization, respectively) problem has a unique solution at the stationary point. In the proof, we show that if ϕ is strongly convex, then the dual value function $\mathfrak{J}$ becomes strongly concave, so that the dual problem has a solution at the stationary point. Therefore, strong convexity of ϕ plays the same role as the two assumptions - weakly monotone ϕ'' and $\mathfrak{J}_2 < 0$ - necessary for existence in Proposition 3. Since the dual convex functional ϕ in Example 4 is indeed strongly convex on $[0, \infty)$, existence follows by Proposition 6.

4.2. Extended Real-valued Dual Convex Functional

We next examine the issue of existence when the dual convex functional ϕ diverges on its domain. Recall that ϕ takes ∞ as a value when u is unbounded below or u is increasing along an oblique asymptote as in Example 3. In this case, we encounter the problem of how to evaluate the expectation of ϕ in the expression (3) of $\mathfrak{J}$, and as a result, the objective of the dual problem is in general not well-defined. In order to circumvent this technical issue, we define the truncated dual convex functional as

$$\phi^M(q) \;=\; \begin{cases} M & \text{for } q \text{ at which } \phi(q) = \infty \\ \phi(q) & \text{otherwise,} \end{cases}$$

and define the associated dual value function $\mathfrak{J}^M$ as

$$\mathfrak{J}^M(\mathbf{z}) \;=\; -\mathbb{E}\left[\phi^M\left(z_1 + z_2 L^a(S)\right) \mid a\right] + z_1(\psi(a) + \underline{V}) + z_2\psi'(a).$$

Then it follows by the Lebesgue dominated convergence theorem that $\lim_{M\to\infty} \mathfrak{J}^M(\mathbf{z}) = \mathfrak{J}(\mathbf{z})$.

We split analysis into two cases depending on whether the likelihood ratio L^a is bounded or not. When unbounded, our dual approach yields the next well-known result:

Proposition 7 (Unbounded Likelihood Ratios, Mirrlees (1979) and Holmström (1977))**.** *Suppose that an agency problem $\langle u, \psi, F\rangle \in \Omega$ gives rise to an extended real-valued dual convex functional ϕ. If the likelihood ratio L^a of F is unbounded, there exists no optimal contract implementing the desired action a.*

PROOF OF PROPOSITION 7: See Appendix A.5. $\square$

Proposition 7 formalizes the original idea of nonexistence developed by Mirrlees (1979) in terms of the dual functional. When the agent's utility function and the likelihood ratio are unbounded below, Mirrlees has shown that there exists a sequence of simple contracts enforcing severe penalties for extremely poor performance, approximating the first-best outcome arbitrarily closely.[10] Our dual approach embodies this classic result in the dual space. For an illustration, we first demonstrate in the proof that given the conditions, the dual problem $\max \mathfrak{J}(\mathbf{z})$ has a unique solution at the *corner* where $z_2 = 0$. However, $z_2 = 0$ implies that the incentive constraint is no longer concern as in the environment with perfect information, and as a result, the problem yields exactly the first-best value. It then follows by strong duality that if the primal problem is feasible so that $\mathfrak{C}(a) < \infty$, the *infimum* value $\mathfrak{C}(a)$ also takes the same value. In addition, if some contract achieves this value, it should take a form of fixed pay, namely $w_{\mathbf{z}}$ with $z_2 = 0$, thereby violating the incentive constraint. To recapitulate, this falls into the case where the infimum value $\mathfrak{C}(a)$ is not an element of the set of feasible values. Therefore, we can deduce that there exists a sequence of feasible contracts $\{w_n\}$ such that $\mathbb{E}[w_n|a] \to \mathfrak{C}(a)$ but its limit point is infeasible.

A simple structure of the dual problem also carries the exact conditions on the primitives under which the dual value function $\mathfrak{J}$ achieves the optimal value at $z_2 = 0$ and hence a first-best approximation is possible. Notice that strong duality is indispensable for Proposition 7. In order to establish strong duality, we employ the weak condition (ii) in Theorem 1 that deserves comment. To highlight its role, suppose that the agent has a utility function unbounded below but bounded above due to limited liability for the principal. In this case, it is unclear whether the given agency problem satisfies other existing conditions for non-degeneracy, which require that both constraints should have slack.[11] In contrast, the condition (ii) only requires that either constraint should have enough slack to offset the other. Compared to Slater's condition, it can be easily verified even when the dual value function $\mathfrak{J}$ is not well-behaved as in Proposition 7 or incentives are bounded.

Lastly we turn to the remaining case, where the dual convex functional ϕ takes ∞ on its domain and the underlying distribution entails a *bounded* likelihood ratio. We know that ϕ is an extended real-valued function, when u is either unbounded below or increasing with an oblique asymptote. In contrast with Proposition 7, however, we obtain qualitatively different existence results in these two cases, so we treat them separately.

In the agency model, the bounded ratio indicates that the given performance measure $\mathbf{X}$ conveys limited information about the agent's behavior. As a result, no longer can the principal incentivize the agent with severe penalties or big bonuses for an unexpected outcome. In our dual approach, the bounded ratio makes a noteworthy difference from Proposition 7. Despite the dual convex functional ϕ taking the value ∞, the dual value function $\mathfrak{J}$ can take a finite value even at a positive z_2, provided that the other multiplier z_1 is adjusted accordingly. To be precise, for bounded L^a, we can keep the random variable $z_1 + z_2 L^a(S)$ from penetrating the region where $\phi = \infty$. This suggests that the dual problem has an interior solution with $z_2 > 0$, precluding the first-best approximation. Nonetheless, as is shown by Moroni and Swinkels (2014), an optimal contract

[10]In the same vein, Holmström (1977) has shown that if the agent's utility has an oblique asymptote, then the first-best outcome can be approximated by a sequence of contracts paying high bonus for extremely good performance.

[11]A standard way to examine this *constraint qualification* condition in the agency model is to construct a simple contract consisting of a fixed wage and a bonus for incentives, and then check whether the scheme satisfies both constraints. However, whenever utility or payment is bounded as in Example 3, the standard method is not well-suited because such a simple contract may violate either constraint.

fails to exist for a different reason when the agent has a utility function unbounded below.

Proposition 8 (Bounded Likelihood Ratios, (Moroni and Swinkels, 2014))**.** *Consider an agency problem $\langle u, \psi, F \rangle \in \Omega$, in which the agent's utility function u is unbounded below and the likelihood ratio of F is bounded: $L^a \in [\underline{L}, \overline{L}]$ almost everywhere for intended action a. Then there exists no optimal contract implementing the action a, provided that*

$$\inf \{\mathfrak{J}_2(\mathbf{z}) : z_1 \geq -z_2 \underline{L},\, z_2 \geq 0\} \;>\; 0. \tag{10}$$

Proposition 8 presents a sufficient condition for nonexistence in terms of the dual value function $\mathfrak{J}$. To illustrate the key insight into the condition (10), letting a be a target action and $\underline{L} \equiv \inf_{s \in \mathcal{S}} L^a(s) \in (-\infty, 0)$, consider the dual problem $\max \mathfrak{J}(\mathbf{z})$. As we have seen in the proof of Proposition 7, the objective function $\mathfrak{J}$ tends to $-\infty$ whenever the variable $z_1 + z_2 L^a(S)$ falls below zero with positive probability. This suggests that in order to maximize $\mathfrak{J}$ given $z_2 > 0$, the multiplier z_1 has to be set at least larger than $-z_2 \underline{L}$. Consequently, whenever the utility function is unbounded below, the bounded ratio imposes a restriction on z_1 in the dual problem. With this feature in hand, suppose (10) holds, so that the function $\mathfrak{J}$ is continuously increasing in z_2. Then we have $\mathfrak{J}$ arbitrarily large as z_2 grows, and thus weak duality (wD) leads us to $\mathfrak{C}(a) = \infty$, that is, the primal problem is infeasible.

Given the conditions on u and L^a in Proposition 8, Moroni and Swinkels (2014) have presented a concrete example for nonexistence in which the agent has a logarithmic utility function, $u(w) = \ln w$ defined on $(0, \infty)$. Recall that its dual convex functional takes a form of

$$\phi(q) \;=\; \begin{cases} q(\ln q - 1) & \text{for} \quad q \geq 0, \\ \infty & \text{for} \quad q < 0. \end{cases}$$

Since the functional ϕ has a second-order derivative decreasing in q for $q > 0$, it follows from the Banks (1963) inequality (Refer to Appendix B) that

$$\mathfrak{J}_{12} \;=\; -\mathbb{E}\left[\phi''\left(z_1 + z_2 L^a(S)\right) L^a(S) | a\right] \;\geq\; 0.$$

To put it in words, an increase in the level of pay z_1 brings about a perverse effect on the agent's marginal incentive $\mathfrak{J}_2$. Consequently, for each $z_2 > 0$, the infimum in (10) is attained at $z_1 = -z_2 \underline{L}$. To see whether the infimum of $\mathfrak{J}_2$ is strictly larger than zero, we take the derivative of $\mathfrak{J}_2(-z_2 \underline{L}, z_2)$ with respect to z_2, to obtain

$$\frac{d}{dz_2} \mathfrak{J}_2(-z_2 \underline{L}, z_2) \;=\; -\frac{d}{dz_2} \int \log(-z_2 \underline{L} + z_2 L^a(s)) f_a(s|a) ds \;=\; -\frac{1}{z_2} \int f_a(s|a) ds \;=\; 0. \tag{11}$$

This implies that the strongest marginal incentive is invariant with respect to z_2, and therefore $\inf \mathfrak{J}_2 > 0$ holds if the assigned task incurs sufficiently high marginal costs to the agent.

To develop insights into this negative result, remember that an unbounded-below utility function imposes a lower bound on the multiplier z_1, which is needed to protect the agent against undue penalties. The key to nonexistence in Proposition 8 lies in the fact that the lower bound $-z_2\underline{L}$ is directly proportionate to z_2, which controls the pay-performance sensitivity in the contract $w_{\mathbf{z}}$. As a result, an increase in z_2 induces a countervailing effect on incentive provision. A contract with high sensitivity does not just strengthen incentive to work but tightens the lower bound on base pay z_1. Hence the principal cannot help but raise the base pay, which undermines the incentives as is indicated above by $\mathfrak{J}_{12} \geq 0$. As a matter of fact, (11) reveals that in case of log utility, these two opposite effects are *exactly* offset, and therefore the principal can no longer provide adequate incentives via a dual contract $w_{\mathbf{z}}$.

Our next result presents a rather simple condition for bounded incentives (10) in terms of the dual convex function alone:

Corollary 9. *Consider an agency problem $\langle u, \psi, F\rangle \in \Omega$, in which the agent's utility function u is unbounded below and the likelihood ratio L^a of F is bounded. If the associated dual convex function satisfies $\lim_{q\uparrow\infty} \phi'(q) = \kappa$ for some constant κ, there exists no optimal contract implementing the desired action.*

Proof of Corollary 9: For the proof, we claim that the dual problem achieves an infinite value under the given conditions. To this end, we examine the limiting behavior of $\mathfrak{J}_2$ at $\mathbf{z}' = (-z_2\underline{L}, z_2)$ as $z_2 \to \infty$:

$$\begin{aligned} \lim_{z_2\uparrow\infty} \mathfrak{J}_2(\mathbf{z}') &= -\mathbb{E}\left[\lim_{z_2\uparrow\infty} \phi'\left(z_2(L^a(S) - \underline{L})\right) L^a(S)|a\right] + \psi'(a) \\ &= -\mathbb{E}\left[\kappa L^a(S)|a\right] + \psi'(a) \;>\; 0, \end{aligned}$$

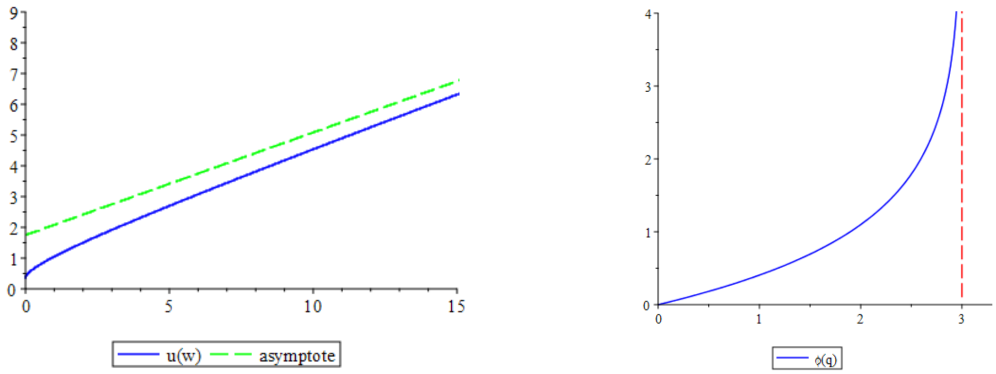
where the first equality is due to the dominated convergence theorem and the second is due to the fact that $\phi' \to \kappa$ almost surely. The obtained inequality $\mathfrak{J}_2 > 0$, due to $\mathbb{E}[L^a(S)|a] = 0$, implies that moving in the direction of $(-\underline{L}, 1)$ continuously, the dual value function $\mathfrak{J}$ tends to ∞. This is sufficient for $\mathfrak{J}$ to grow arbitrarily large, so that $\mathfrak{C}(a) = \infty$ follows by weak duality. □

For an application, suppose that the agent's utility function exhibits constant relative risk aversion (CRRA), $u(w) = \frac{1}{1-\gamma} w^{1-\gamma}$ on $(0, \infty)$, where $\gamma > 1$ represents the relative coefficient. Its dual convex function ϕ takes a form of

$$\phi(q) \;=\; \begin{cases} \frac{\gamma}{1-\gamma} q^{1/\gamma} & \text{for} \quad q \geq 0, \\ \infty & \text{for} \quad q < 0, \end{cases}$$

so that $\lim_{q\uparrow\infty} \phi'(q) = 0$. Hence existence fails by Corollary 9. Notice that in contrast to the log-utility case, the nonexistence result for CRRA is independent of other elements of the agency model, such as the distribution F of outcomes and the marginal cost $\psi'(a)$ from the assigned task.

From the last two propositions, we can deduce that the agent's bounded-below utility is vital for existence of an optimal contract. Whenever utility is unbounded below, existence is likely to fail irrespective of whether the likelihood ratio is unbounded, i.e., how informative the avail-



(a) $u(w) = \inf_{q>0}\{q^{-1}(\phi(q) + w)\}$ (b) $\phi(q) = \sup_{w\geq 0}\{u(w)q - w\}$

Figure 3: Utility Function with an Oblique Asymptote

able performance measure is. However, it is worthwhile to note that the underlying reason for nonexistence is different in nature, and our dual approach helps clarify the distinction. No feasible contracts can achieve the first-best value in the case of unbounded likelihood ratios, whereas no cost-minimizing contracts can induce the desired action from the agent in the case of bounded ratios.

Our final result of this section concerns the case where the agent's utility function is continuously increasing along an oblique asymptote, $\lim_{w\to\infty} u'(w) = 1/\kappa$. In sharp contrast to Proposition 7 and 8, we can show that for this type of function u, the bounded likelihood ratio leads to existence under a certain condition. A key to this positive result lies in the fact that that the dual convex functional of u now imposes an *upper* bound on the multiplier z_1. Therefore, when providing an incentive through the contract $w_{\mathbf{z}}$, no longer does the principal have to increase the level of pay z_1 in parallel with z_2, thereby nullifying the countervailing effect on the marginal incentive.

Proposition 10. *Consider an agency problem $\langle u, \psi, F\rangle \in \Omega$, in which the agent's utility function u is bounded below and increasing along an oblique asymptote, $\lim_{w\to\infty} u'(w) = 1/\kappa$ for some constant $\kappa > 0$, and the likelihood ratio L^a of F is bounded. Then an optimal contract exists, provided that ϕ'' is weakly monotone and*

$$\int_d^\kappa \phi'(q)dq \;=\; \infty \quad \textit{for some } d < \kappa. \tag{12}$$

Proof of Proposition 10: See Appendix A.6. □

Example 5. *Suppose the agent's utility function gives rise to a dual convex functional taking the form of*

$$\phi(q) \;=\; \begin{cases} 0 & \textit{for } \; q < 0 \\ \ln\frac{\kappa}{\kappa - q} & \textit{for } \; 0 \leq q < \kappa, \end{cases}$$

and $\phi(q) = \infty$ for $q \geq \kappa$, as is depicted in Figure 3-(b) *for $\kappa = 3$. It can be shown that the associated* *function u is derived by the inverse transform of* T *introduced in* Example 4 *and takes a form of*

$$u(w) \;=\; \frac{1}{q^*}\left[\ln\frac{\kappa}{\kappa-q^*}+w\right],$$

where q^* is implicitly defined by

$$w-\ln\frac{q^*}{\kappa-q^*} \;=\; \frac{q^*}{\kappa-q^*}.$$

As is displayed in Figure 3-(a)*, the derived function u is bounded below, strictly concave, and increasing along the oblique asymptote $1/\kappa$. Nonetheless, as the given functional ϕ meets the two conditions of* Proposition 10*, an optimal contract exists so long as the likelihood ratio of F is bounded.*

5. Conclusion

In this paper, we developed a dual approach to the agency problems under the assumption that the agent's incentive constraints can be replaced with a single local constraint, i.e., the first-order approach is valid. We demonstrated that by formulating its dual problem, the principal's cost-minimization problem for implementation of an action from the agent can be analyzed in a systematic manner. In particular, the dual formulation separates the problem of finding an optimal contract from the problem of identifying the Lagrange multipliers, thereby investigating the existence issue in a compact way and offering illuminating insights into the previous nonexistence examples. Furthermore, as we show in the companion paper (Chi and Choi (2021)), the approach enables us to characterize a more efficient performance measure in the principal-agent problem. We hope that the methods developed in this paper can serve as a useful tool to study more enriched agency model with other constraints on feasible contracts such as limited liability or commitment, which we leave for future work.

A. Omitted Proofs

A.1. Proof of Theorem 1

We follow Luenberger (1969) to establish strong duality with the relaxed condition (ii) for non-degeneracy. To start, let $\mathbf{z} = (z_1, z_2) \in \Re^2$ and define the following two subsets of $\Re^3$ as

$$\begin{aligned} \Xi^1 &= \{(r, \mathbf{z}) : r \geq \mathbb{E}[w(S)|a],\ z_1 \geq \underline{V} - V(w, a),\ z_2 \geq -V_a(w, a) \text{ for some } w \geq \underline{w}\} \\ \Xi^2 &= \{(r, \mathbf{z}) : r \leq \mathfrak{C}(a),\ \mathbf{z} \leq \mathbf{0}\}. \end{aligned}$$

Observe that Ξ^1 and Ξ^2 both are a convex set. In addition, it follows by definition of $\mathfrak{C}(a)$ that Ξ_1 contains no interior points of Ξ^2, whereas Ξ^2 has an interior point of Ξ^1 if $\mathfrak{C}(a) < \infty$. Then by the separating hyperplane theorem, there exists a nonzero vector $(r^*, \mathbf{z}^*) \in \Re^3$ separating the two convex sets, that is,

$$(r^*, \mathbf{z}^*) \cdot (r^1, \mathbf{z}^1) \ \geq \ (r^*, \mathbf{z}^*) \cdot (r^2, \mathbf{z}^2) \tag{A.1}$$

for all $(r^1, \mathbf{z}^1) \in \Xi^1$ and $(r^2, \mathbf{z}^2) \in \Xi^2$.

It is immediate from the definition of the set Ξ^2 that $(r^*, \mathbf{z}^*) \geq \mathbf{0}$, for otherwise the inner product on the right-hand side of (A.1) would range up to positive infinity so the inequality does not always hold. Furthermore, since the point $(\mathfrak{C}(a), \mathbf{0})$ is an element of Ξ^2, we have

$$r^* r^1 + (\mathbf{z}^*, \mathbf{z}^1) \ \geq \ r^* \mathfrak{C}(a) \qquad \forall\ (r^1, \mathbf{z}^1) \in \Xi^1. \tag{A.2}$$

We next prove $r^* > 0$. To this end, suppose to the contrary that $r^* = 0$. Then $\mathbf{z}^* \neq \mathbf{0}$, and substituting $\mathbf{z} = (\underline{V} - V(w, a), -V_a(w, a))$ into (A.2) gives us

$$z_1^*(\underline{V} - V(w, a)) - z_2^* V_a(w, a) \ \geq \ 0 \quad \text{for all } w \geq \underline{w}.$$

However, the obtained inequality then fails to meet condition (ii) of the theorem, since it requires that the inequality should be reversed for at least one contract w. Consequently, the non-degeneracy condition (ii) leads us to $r^* > 0$, namely, a non-vertical hyperplane separating the two convex sets Ξ^1 and Ξ^2.

To complete the proof, we normalize and substitute $r^* = 1$ into (A.2). This gives us

$$\inf_{(r, \mathbf{z}) \in \Xi^1} \{r + \mathbf{z}^* \cdot \mathbf{z}\} \ \geq \ \mathfrak{C}(a).$$

However, by definition of the set Ξ^1, we must have

$$\begin{aligned} \inf_{(r, \mathbf{z}) \in \Xi^1} \{r + \mathbf{z}^* \cdot \mathbf{z}\} \ &\leq \ \inf_{w \geq \underline{w}} \{\mathbb{E}[w|a] + z_1^*(\underline{V} - V(w, a)) - z_2^* V_a(w, a)\} \ = \ \inf_{w \geq \underline{w}} \mathfrak{L}(w; \mathbf{z}^*) \\ &\leq \ \inf\{\mathbb{E}[w|a] : w \geq \underline{w} \text{ satisfies (PC) and (L-IC)}\} \ = \ \mathfrak{C}(a). \end{aligned}$$

Therefore, the inequalities above must hold with equality, and by definition of the dual value function $\mathfrak{J}$, we obtain $\inf_w \mathfrak{L}(w; \mathbf{z}^*) = \mathfrak{J}(\mathbf{z}^*) = \mathfrak{C}(a)$. Therefore, the desired result $\max_{\mathbf{z} \geq \mathbf{0}} \mathfrak{J}(\mathbf{z}) = \mathfrak{C}(a)$ follows by weak duality (wD). □

A.2. Proof of Lemma 1

For (i), observe that $u(w)q - w$ is monotone decreasing in w for every $q \leq 0$. Hence the dual convex function is given by $\phi(q) = u(\underline{w})q - \underline{w}$ on the interval $(-\infty, 0]$, which is finite-valued if $\lim_{w \downarrow \underline{w}} u(w) > -\infty$. Its contrapositive tells us that if the agent's utility function is unbounded below, then the corresponding dual convex function takes the value ∞, suggesting that ϕ can be an extended real-valued function.

For (ii), define $\bar{q} \equiv \inf\{q > 0 : \exists\, w^* \in [\underline{w}, \infty)$ at which $u'(w) = 1/q\}$. Then for every $q \geq \bar{q}$, the given condition $u'(w) \to 0$ ensures existence of a unique w^* at which $u'(w^*) = 1/q$. Hence ϕ is finite over the interval $[\bar{q}, \infty)$. To complete the proof, we divide analysis into two cases according to whether u is bounded at $\underline{w}$. If u is unbounded below, marginal utility from wealth $u'(w)$ must grow large as $w \downarrow \underline{w}$. This implies $\bar{q} = 0$, so that the function $\phi(q)$ takes a finite value for every $q > 0$. On the other hand, if u is bounded below, $\bar{q}$ can be strictly positive. Even if this is the case, for every $q \in [0, \bar{q})$, $u(w)q - w$ is then decreasing in w and thus $\phi(q) = u(\underline{w})q - \underline{w}$ is finite. Therefore, the function ϕ is finite on $(0, \infty)$ in either case.

For (iii), note that for every $\alpha \in (0, 1)$ and $q_1 \neq q_2$,

$$\begin{aligned} \phi(\alpha q_1 + (1-\alpha) q_2) &= \sup_{w \geq \underline{w}} \{\alpha(u(w)q_1 - w) \;+\; (1-\alpha)(u(w)q_2 - w)\} \\ &\leq \alpha \sup_{w \geq \underline{w}} \{u(w)q_1 - w\} \;+\; (1-\alpha) \sup_{w \geq \underline{w}} \{u(w)q_2 - w\} \\ &= \alpha\phi(q_1) \;+\; (1-\alpha)\phi(q_2). \end{aligned}$$

Furthermore, it is a routine task to check that the weak inequality above holds strictly, provided that the solution $w^*(q) \in \operatorname{argmax}_{w \geq \underline{w}} u(w)q - w$ is an one-to-one mapping: $w^*(q_1) \neq w^*(q_2)$ for every $q_1 \neq q_2$. □

A.3. Proof of Proposition 3

We organize the proof in a succession of steps. Suppose that condition (8) holds, so that for every z_1, $\mathfrak{J}_2 < 0$ as z_2 grows large. Since the partial derivative $\mathfrak{J}_2$ is monotone decreasing in z_2, and since $\mathfrak{J}_2 > 0$ at $z_2 = 0$, the condition ensures existence of a unique $z_2(z_1) \geq 0$ for each z_1 such that $\mathfrak{J}_2(z_1, z_2(z_1)) = 0$.

- Step 1: *For every $z_2 \geq 0$, there exists a unique $z_1(z_2) \geq 0$ at which $\mathfrak{J}_1(z_1(z_2), z_2) = 0$ holds.*

  To prove this statement, recall that we assume existence of the first-best contract for every problem within Ω. In terms of the dual, there exists a $z^F > 0$ at which the associated dual value function $\mathfrak{J}^F(z) = -\phi(z) + z[\psi(a) + \underline{V}]$ attains a global maximum. Since $\mathfrak{J}^F(z)$ is globally concave, z^F is uniquely determined by the first-order condition: $\phi'(z^F) = \psi(a) + \underline{V}$. This allows us to rewrite the partial derivative $\mathfrak{J}_1(z_1, z_2)$ as

$$\mathfrak{J}_1(\mathbf{z}) \;=\; -\mathbb{E}[\phi'(z_1 + z_2 L^a(S)) - \phi'(z^F)|a].$$

Observe that for every $z_2 \geq 0$, $z_1 + z_2 L^a(S) > z_1^F$ almost surely as z_1 grows large. Consequently, $\lim_{z_1 \uparrow \infty} \mathfrak{J}_1(\mathbf{z}) < 0$ because ϕ' is an increasing function. Therefore, the desired result follows from the fact that $\mathfrak{J}_1$ is monotone decreasing in z_1.

- STEP 2: *If ϕ'' is weakly monotone, the implicit functions $z_1(z_2)$ and $z_2(z_1)$ both are weakly monotone.*

  For the second step, we shall prove the fact that a nondecreasing ϕ'' results in $z_2(z_1)$ nonincreasing. The proof below can be easily generlized to show that the statement is indeed true. To this end, fix a $z_1 \geq 0$ and consider the following optimization problem: $\max_{z_2 \geq 0} \mathfrak{J}(z_1, z_2)$. As is shown in the main body, the objective function is strictly concave in the variable z_2 given z_1, the first-order condition $\mathfrak{J}_2 = 0$ characterizes the unique solution in full. This also reveals that the implicitly-defined function $z_2(z_1)$ can be alternatively put as a unique solution to the problem. On the other hand, if ϕ'' is nondecreasing, we have

$$\mathfrak{J}_{12} \;=\; -\mathbb{E}[\phi''(z_1 + z_2 L^a(S)) L^a(S)|a] \;\leq\; 0,$$

  where the inequality is due to Banks (1963) (Refer to Appendix B). However, as $\mathfrak{J}_{12} \leq 0$ implies that the function $\mathfrak{J}(z_1, z_2)$ exhibits decreasing differences in $(z_2; z_1)$, the solution $z_2(z_1) = \operatorname{argmax}_{z_2 \geq 0} \mathfrak{J}(z_1, z_2)$ is nonincreasing.

- STEP 3: *The dual problem has a unique solution.*

  Define a mapping $\sigma : [0, \infty)^2 \to [0, \infty)^2$ as $\sigma(\mathbf{z}) = (z_1(z_2), z_2(z_1))$. Since this mapping is (weakly) monotone as we verified above, it follows by Tarski's fixed point theorem that σ has a fixed point $\mathbf{z}^*$. Since the fixed point of σ constitutes a stationary point of the dual value function $\mathfrak{J}$, and since the function $\mathfrak{J}$ is strictly concave, the dual problem has a unique solution at $\mathbf{z}^*$.

- STEP 4: *An optimal contract exists in a unique form.*

  To establish existence, we first prove that strong duality holds under the given conditions. For $z_1^* > 0$, our discussion in STEP 3 tells us that at the unique solution $\mathbf{z}^*$ to the dual problem, we have $\nabla \mathfrak{J} = \mathbf{0}$. Since both $\mathfrak{J}_{11}$ and $\mathfrak{J}_{22}$ take a negative value on the domain, there exists a $\mathbf{z}'$ in the neighborhood of $\mathbf{z}^*$ at which $\nabla \mathfrak{J}(\mathbf{z}_0) < \mathbf{0}$. However, $\nabla \mathfrak{J}(\mathbf{z}_0) < 0$ implies that both PC and L-IC constraints are slack for contract $w_0 \equiv w_{\mathbf{z}_0}$:

$$V(w_0, a) - \underline{V} \;>\; 0 \quad \text{and} \quad V_a(w_0, a) \;>\; 0,$$

and thus Slater's condition is satisfied. To see $\mathfrak{C}(a) < \infty$, observe that the two constraints are binding at the contract $w^* = w_{\mathbf{z}^*}$, since $\nabla \mathfrak{J}(\mathbf{z}^*) = \mathbf{0}$. In other words, the contract w^* derived from the dual approach is feasible by construction, and hence we have $\mathfrak{C}(a) \leq \mathbb{E}[w_{\mathbf{z}^*}(S)|a] < \infty$. Therefore, it follows by Theorem 1 that strong duality holds, and Corollary 2 guarantees existence of an optimal contract.

Uniqueness of w^* is also immediate from Corollary 2. For an illustration, suppose to the contrary that there exists another optimal contract $\hat{w}$ different from w^* on a set of positive measure. We then must have $\mathfrak{J}(\mathbf{z}^*) = \mathfrak{L}(\hat{w}; \mathbf{z}^*)$. However, the dual convex functional ϕ is characterized by the unique w^* for each point $q = z_1^* + z_2^* L^a(s)$, a contradiction. $\square$

A.4. Proof of Proposition 6

We below show that if the dual convex functional ϕ is finite-valued on the domain and strongly convex on $[0, \infty)$, the dual value function $\mathfrak{J}$ is strongly concave. That is, there exists a constant $m > 0$ such that

$$\begin{bmatrix} \mathfrak{J}_{11} + m & \mathfrak{J}_{12} \\ \mathfrak{J}_{12} & \mathfrak{J}_{22} + m \end{bmatrix} \quad \text{is negative semidefinite.} \tag{A.3}$$

Suppose that $\phi'' > m$ on $[0, \infty)$ for some constant $m > 0$. Then for every $\mathbf{z} \geq \mathbf{0}$, we have

$$\begin{aligned} \mathfrak{J}_{11}(\mathbf{z}) + m \;&=\; -\mathbb{E}\left[\phi''(q_{\mathbf{z}}(S)) \,|\, a\right] + m \;<\; 0, \\ \mathfrak{J}_{22}(\mathbf{z}) + m\sigma^2 \;&=\; -\mathbb{E}\left[\phi''(q_{\mathbf{z}}(S))\,(L^a(S))^2 \,|\, a\right] + m\sigma^2 \;<\; 0, \end{aligned}$$

where σ^2 indicates variance of the likelihood ratio.

We divide our analysis into two cases depending on whether $\sigma^2 \geq 1$. Suppose $\sigma^2 \geq 1$, so that $\mathfrak{J}_{22} + m \leq \mathfrak{J}_{22} + m\sigma^2 < 0$. In this case, the cross-partial derivative of $\mathfrak{J}$ is bounded above by

$$\begin{aligned} (\mathfrak{J}_{12})^2 \;&=\; \left(\mathbb{E}\left[(\phi''(q_{\mathbf{z}}(S)) - m) L^a(S)|a\right]\right)^2 \\ &\leq\; \mathbb{E}\left[(\phi''(q_{\mathbf{z}}(S)) - m)|a\right] \mathbb{E}\left[(\phi''(q_{\mathbf{z}}(S)) - m)\,(L^a(S))^2\,|a\right] \\ &=\; (\mathfrak{J}_{11} + m)(\mathfrak{J}_{22} + m\sigma^2) \\ &\leq\; (\mathfrak{J}_{11} + m)(\mathfrak{J}_{22} + m), \end{aligned} \tag{A.4}$$

where the first equality results from $\mathbb{E}\left[L^a(\mathbf{X})|a\right] = 0$ and the first inequality is due to the Cauchy-Schwarz inequality. Consequently, the matrix in (A.3) is negative semidefinite.

In case of $\sigma^2 < 1$, we let $\mu \equiv m\sigma^2 > 0$ so that

$$\mathfrak{J}_{11}(\mathbf{z}) + \frac{\mu}{\sigma^2} \ < \ 0 \quad \text{and} \quad \mathfrak{J}_{22}(\mathbf{z}) + \mu \ < \ 0.$$

Consequently, $\mathfrak{J}_{11}(\mathbf{z}) + \mu < \mathfrak{J}_{11} + \mu\sigma^{-2} < 0$. With this in hand, substituting $\mu\sigma^{-2}$ for m in (A.4) yields

$$(\mathfrak{J}_{12})^2 \ \leq \ (\mathfrak{J}_{11} + \mu\sigma^{-2})(\mathfrak{J}_{22} + \mu) < (\mathfrak{J}_{11} + \mu)(\mathfrak{J}_{22} + \mu).$$

This proves that the matrix in (A.3) with $m = \mu$ is negative semidefinite for all σ^2, and hence the objective function $\mathfrak{J}$ of the dual problem is strongly concave. Furthermore, since a strongly concave function (so *coercive*) defined on a closed set has a maximizer (Corollary 11.17 in Bauschke and Combettes (2011)), strong concavity of $\mathfrak{J}$ guarantees existence of a solution to the dual problem. Existence of an optimal contract is then immediate from strong duality as in Proposition 3. The proof is now complete. □

A.5. Proof of Proposition 7

We first show that under the given conditions, the dual value function $\mathfrak{J}$ takes $-\infty$ as a value: to be precise, $\lim_{M\to\infty} \mathfrak{J}^M(\mathbf{z}) = -\infty$ whenever $z_2 > 0$.

Suppose that the utility function u is unbounded below, so that the dual convex functional ϕ takes ∞ for $q < 0$. We break up the truncated function $\mathfrak{J}^M$ into two pieces, to write

$$\begin{aligned}\mathfrak{J}^M(\mathbf{z}) \ = \ -M \cdot \Pr\left(z_1 + z_2 L^a(S) < 0 | a\right) - \mathbb{E}\left[\phi\left(z_1 + z_2 L^a(S)\right) \mathbb{1}_{\{z_1 + z_2 L^a(S) \geq 0\}} \,|\, a\right] \\ + z_1(\psi(a) + \underline{V}) + z_2 \psi'(a).\end{aligned} \tag{A.5}$$

Observe that whenever the ratio L^a is unbounded, the event $\{s \in \mathcal{S} : z_1 + z_2 L^a(s) < 0\}$ occurs with positive probability for every $z_2 > 0$. Hence the first term in (A.5) tends to $-\infty$ as M grows large, whereas the other terms are independent of M. Moreover, since ϕ is finite-valued in the region $[0, \infty)$ and thus is convex by Lemma 1, we apply Jensen's inequality to obtain

$$\mathbb{E}\left[\phi\left(z_1 + z_2 L^a(S)\right) \mathbb{1}_{\{z_1 + z_2 L^a(S) \geq 0\}} | a\right] \ \geq \ \phi\Big(\mathbb{E}\left[\max\left\{z_1 + z_2 L^a(S), 0\right\} \,|\, a\right]\Big) \ > \ -\infty.$$

Hence the function $\mathfrak{J}^M$ diverges to $-\infty$ for every $z_2 > 0$.

On the other hand, the dual value function $\mathfrak{J}$ takes a finite value at $z_2 = 0$ regardless of z_1. This tells us that a solution to the dual problem, if it exists, must occur at the corner $(z_1, 0)$. When $z_2 = 0$, however, the function $\mathfrak{J}$ boils down to the same form as the one under perfect information. Consequently, the dual problem has a unique solution at $(z^F, 0)$, where $z^F > 0$ indicates the unique solution to the first-best dual problem. In case of the utility function increasing along an oblique asymptote, $\lim_{w\to\infty} u'(w) = 1/\kappa$, the associated functional ϕ tends to ∞ on $[\kappa, \infty)$. So if L^a is unbounded (above), for every $z_2 > 0$, the variable $z_1 + z_2 L^a(S)$ would traverse the region $[\kappa, \infty)$ with positive probability. Therefore, $\mathfrak{J}$ attains a global maximum at the same point. We omit the detail.

Based on the structure of the dual problem, we next prove non-existence of an optimal contract in the primal problem. In line with the previous results, the key is strong duality which creates a link between the two problems. To proceed, given a pair of multipliers $\mathbf{z} > \mathbf{0}$, let $w_{\mathbf{z}}$ denote the contract characterizing the dual convex functional ϕ. Rewriting $\mathfrak{J}$ in terms of $w_{\mathbf{z}}$, we obtain

$$\mathfrak{J}(\mathbf{z}) \;=\; \mathfrak{L}(w_{\mathbf{z}};\mathbf{z}) \;=\; \mathbb{E}[w_{\mathbf{z}}(S)|a] + z_1\left(\underline{V} - V(w_{\mathbf{z}},a)\right) - z_2 V_a(w_{\mathbf{z}},a). \tag{A.6}$$

Observe that since $w_{\mathbf{z}} \geq \underline{w}$, the first term of $\mathfrak{J}$ in (A.6) is bounded below regardless of $\mathbf{z}$. Nevertheless, as is shown above, the expression $\mathfrak{J}$ takes $-\infty$ whenever $z_2 > 0$. This implies that $z_1(\underline{V} - V(w_{\mathbf{z}},a)) - z_2 V_a(w_{\mathbf{z}},a) < 0$ for every $\mathbf{z} > \mathbf{0}$, and therefore the non-degeneracy condition (ii) in Theorem 1 is met.

To complete the proof, suppose to the contrary that there is a feasible contract $\hat{w}$ that achieves the constrained minimum: $C(a) = \mathbb{E}[\hat{w}(S)|a]$. Note that this is sufficient for the primal problem to be feasible, and thus $\mathfrak{C}(a) < \infty$. Then strong duality holds by Theorem 1, so we have $C(a) = \mathfrak{J}(z^F, 0)$. Put it another way, the contract $\hat{w}$ results in the first-best value, and consequently, $\mathfrak{J}(z^F, 0) = \mathfrak{L}(\hat{w}; z^F, 0)$ by definition. This suggests that $\hat{w}$ should be equal almost everywhere to $w_{z^F,0} = w^*(z^F)$, the contract which uniquely determines the dual value. However, the latter promises to the agent fixed pay regardless of outcomes, violating the incentive constraint. Therefore, the constrained minimum is never achieved by any feasible contracts. □

A.6. Proof of Proposition 10

Given an intended action a, denote by $\overline{L} \equiv \sup_s L^a(s) \in (0,\infty)$ the least upper bound for the likelihood ratio. Given the condition on u, the associated dual convex functional ϕ takes the value ∞ on the interval $(\kappa,\infty]$. Hence, in order to maximize $\mathfrak{J}$ given $z_2 > 0$, we must have $z_1 \leq -\mu\overline{L} + \kappa$ so that $q_{\mathbf{z}}(s) \leq \kappa$ for all possible scores s, for otherwise the function $\mathfrak{J}$ would tend to $-\infty$. Consequently, the dual problem $\max_{\mathbf{z}\in\Gamma} \mathfrak{J}(\mathbf{z})$ becomes constrained over compact set $\Gamma = \{\mathbf{z} \geq \mathbf{0} : z_1 + z_2\overline{L} \leq \kappa\}$. Since the objective function is continuous on this set, it follows by the extreme value theorem that the dual problem possesses a solution $\mathbf{z}^*$ in Γ.

We next follow the proof of Proposition 3 to complete the proof. It remains to show that the given conditions on ϕ implies $\mathfrak{J}_2(\mathbf{z}^*) = 0$. To this end, observe that for every $\mathbf{z} \in \Gamma$, we have

$$\mathfrak{J}_2(\mathbf{z}) \;=\; -\mathbb{E}\left[\phi'(q_{\mathbf{z}}(S))L^a(S)\mathbb{1}_{\{q_{\mathbf{z}}<d\}}\right] - \mathbb{E}\left[\phi'(q_{\mathbf{z}}(S))L^a(S)\mathbb{1}_{\{d<q_{\mathbf{z}}\leq\kappa\}}\right] + \psi'(a).$$

In this expression, the first term is bounded for every $\mathbf{z}$, whereas the second term grows arbitrarily large, due to the condition (12), as $\mathbf{z}$ is close to the boundary of the set Γ so that $z_1 + z_2\overline{L} \to \kappa$. This proves that the solution $\mathbf{z}^*$ should be an interior point of Γ, and therefore, $\nabla\mathfrak{J}(\mathbf{z}^*) = \mathbf{0}$ follows. Consequently, the contract $w_{\mathbf{z}^*}$ obtained from the dual approach is feasible, and existence is thus guaranteed by Corollary 2. □

B. The Banks Inequality

The next inequality is due to Banks (1963), which is widely adopted in literature to establish comparative statics results (e.g., Quah and Strulovici (2009)).

Lemma B.1. *Let L be a measurable real-valued function defined on a measure space $(\mathcal{S}, \mathcal{F}, \nu)$, and let $\varphi : \Re \to [0, \infty)$ be a nondecreasing function. If $\int_{\mathcal{S}} L(s)d\nu(s) = 0$, then $\int_{\mathcal{S}} \varphi(L(s))L(s)d\nu(s) \geq 0$.*

PROOF OF LEMMA B.1: For each $\kappa \in \Re_+$, define $A(\kappa) = \{s \in \mathcal{S} : \varphi(L(s)) \geq \kappa\}$. Then by Fubini's Theorem, we have

$$\int_{\mathcal{S}} \varphi(L(s))L(s)d\nu(s) \;=\; \int_0^\infty \left(\int_{A(\kappa)} L(s)d\nu(s) \right) d\kappa.$$

Since φ is nondecreasing, the set $A(\kappa)$ takes a form of $A(\kappa) = \{s \in \mathcal{S} : L(s) \geq \kappa'\}$ for some κ'. Then the result follows from the fact that $\int_{\mathcal{S}} L(s)d\nu(s) = 0$ implies $\int_{A(\kappa)} L(s)d\nu(s) \geq 0$ for every $\kappa \in [0, \infty)$. $\square$

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