Combining sign and parametric restrictions in SVARs by Givens Rotations

Lance A. Fisher and Hyeon-seung Huh

Abstract

This paper develops a method for combining sign and parametric restrictions in SVARs by means of Givens matrices. The Givens matrix is used to rotate an initial set of orthogonal shocks in the SVAR. Parametric restrictions are imposed on the Givens matrix in a manner which utilises its properties. This gives rise to a system of equations which can be solved recursively for the 'angles' in the constituent Givens matrices to enforce the parametric restrictions. The method is applied to several identifications which involve a combination of sign restrictions, and long-run and/or contemporaneous restrictions in Peersman's (2005) SVAR for the US economy. The method is compared to the recently developed method of Aries, Rubio-Ramírez and Waggoner (2018) which combines zero and sign restrictions.

Key Words: structural vector autoregressions, sign and parametric restrictions, Givens rotations, QR decomposition

JEL Classification: C32, C51, E32

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1. Introduction

The sign restrictions approach to separating the shocks in a structural vector-autoregression (SVAR) into economically interpretable shocks has become increasingly popular. The sign restrictions approach was first introduced by Faust (1998), Canova and De Nicoló (2002) and Uhlig (2005). It involves generating many sets of impulse responses and judging whether each set should be retained or rejected on the basis of the signs of the responses. The retained responses are summarised from which conclusions are drawn. The expected signs of the responses i.e. the sign restrictions typically come from a consensus view of the effects of certain shocks on the economic variables. Traditionally, the shocks in a SVAR have been identified by parametric restrictions. These may take the form that a certain shock has a zero contemporaneous effect on a certain variable (a contemporaneous restriction) or that a certain shock has a zero long-run effect on a certain variable (a long-run restriction). Long-run zero restrictions are typically motivated by long-run neutrality propositions in economics and contemporaneous zero restrictions are often motivated by the notion that economic agents can only observe data on key economic variables with a one period delay (due to information lags).

In this paper, we develop a method to combine sign and parametric restrictions in a SVAR. This method involves rotating an initial set of orthogonal shocks in a SVAR by means of a Givens rotation matrix. The parametric restrictions are imposed on the Givens rotation matrix in a particular way, utilizing the properties of the constituent Givens matrices. The restrictions can be both contemporaneous and long-run, and be zero or non-zero restrictions. Although we apply the method to a 4-variable SVAR, it can be applied to a SVAR of any dimension. The properties of the Given rotation matrix are such that when the parametric restrictions are imposed on its constituent matrices, the resulting set of equations can be solved recursively for the unknowns (the 'angles' in the constituent Givens matrices) that will enforce the parametric restrictions. Baumeister and Benati (2013) imposed a single zero contemporaneous restriction in a sign restrictions framework that utilised the Givens matrix and their approach was extended by Haberis and Sokol (2014) to multiple zero contemporaneous restrictions. We fully develop the method of combining parametric and sign restrictions using Givens rotation matrices in this paper. We relate this method to the recent method of Aries, Rubio-Ramírez and Waggoner (2018) who impose the parametric restrictions on the columns of the rotation matrix generated by the Gram-Schmidt procedure. Their approach can only impose zero restrictions whereas the method which utilises Givens rotations can accommodate non-zero restrictions. Both methods share the characteristic that in a SVAR of *N* variables, at most *N-1* parametric restrictions can be placed on the first shock, *N-2* on the second shock and so forth.

The method is developed in the context of the four variable SVAR of Peersman (2005). Peersman identifies his SVAR first by using parametric restrictions alone and then by using sign restrictions alone. In this paper, we show first how to impose the full set of parametric restrictions which were chosen by Peersman using Givens rotation matrices. This is instructive because it demonstrates two key features of the procedure. The first is that the resulting set of equations is recursive so that the free parameters in the constituent Givens matrices can be solved for to enforce the parametric restrictions. The second is that the rank condition for parametric identification of the SVAR is apparent which is that the shocks for any order must have 3-2-1-0 restrictions placed on them. We then turn to full identification of the shocks in Peersman's model by sign restrictions alone using Givens rotations. We generate draws of the Givens matrix using a different method to Peersman's method. We show that the method we utilise will generate a Givens matrix which is equivalent to an orthogonal matrix produced in a QR decomposition. Having developed the method for the two polar cases, we then develop it for the case of primary interest, namely, for sign restrictions used in combination with parametric restrictions. The method is used to implement five identifications in Peersman's model which involve a combination of sign and parametric restrictions.

The structure of the paper is as follows. Section 2 describes the structural model of Peersman. Section 3 discusses the properties of Givens matrices. Section 4 implements the full parametric identification of Peersman's model by utilising Givens rotation matrices. Section 5 implements the full sign restrictions approach where the constructed Givens matrix is equivalent to the orthogonal matrix from a QR decomposition. Section 6 presents the method to combine sign and parametric restrictions. Section 7 presents the results from five identification schemes which use a combination of sign and long-run and/or contemporaneous zero restrictions in Peersman's model. Section 8 describes the method of Aries, Rubio-Ramírez and Waggoner (2018) for one of our identifications and relates it to the Givens approach. Finally, section 9 concludes.

2. The structural model

The methods in this paper are applied to the four-variable vector-autoregressive (VAR) model of Peersman (2005). In Peersman's model, the variables are the price of oil $(o_t),$ output $(y_t),$ consumer prices (p_t) and the short-term interest rate $(i_t).$ Peersman treats the oil price, output and consumer prices as I(1) variables and the interest rate as an I(0) variable. Let z_t be the vector of variables that enter the VAR. The I(1) variables are differenced once before entering the VAR so that $z_t = \begin{pmatrix}\Delta o_t & \Delta y_t & \Delta p_t & i_t\end{pmatrix}'$. The reduced-form vector moving average representation is:

$$z_t = D(L)e_t \tag{1}$$

where $D(L) = (I + D_1L + D_2L^2 + \ldots),$ L is the lag operator and $e_t \sim (0,\Omega)$. The reduced-form errors (e_t) are orthogonalized by finding a matrix, $A_0^{-1},$ such that $\Omega = A_0^{-1}(A_0^{-1})'.$ In this paper, the matrix A_0^{-1} is obtained from an eigenvalue decomposition of $\Omega,$ but it could also be obtained from a Cholesky decomposition in which case it is lower triangular. This gives:

$$z_t = D(L)A_0^{-1}A_0e_t \tag{2}$$

The orthogonalized shocks (A_0e_t) are rotated using the Givens rotation matrix $G,$ which has the property that $GG' = G'G = I,$ to form a new set of orthogonalized shocks. The matrix G is (4x4) as the model has four variables. Utilising the Givens matrix in Eq. (2), it can be written as:

$$z_t = D(L)A_0^{-1}GG'A_0e_t \tag{3}$$

and the new set of orthogonal shocks is $\varepsilon_t = G'A_0e_t.$ The impulse responses to the new set of orthogonal shocks is given by:

$$C(L) = D(L)A_0^{-1}G \tag{4}$$

The long-run response of the variables to the new orthogonal shocks is given by:

$$C(1) = D(1)A_0^{-1}G \tag{5}$$

The contemporaneous response of the variables to the new orthogonal shocks is given by:

$$C(0) = A_0^{-1}G \tag{6}$$

since $D(0) = I$. We now turn to the properties of the Givens rotation matrices.

3. Properties of Givens matrices

In the four variable case, there are six Givens matrices. They are $G(\theta_{12})$, $G(\theta_{13})$, $G(\theta_{14})$, $G(\theta_{23})$, $G(\theta_{24})$ and $G(\theta_{34})$. They are formed by taking the (4x4) identity matrix and setting $G^{ii}(\theta_{ij}) = \cos\theta_{ij}$, $G^{ij}(\theta_{ij}) = -\sin\theta_{ij}$, $G^{ji}(\theta_{ij}) = \sin\theta_{ij}$, $G^{jj}(\theta_{ij}) = \cos\theta_{ij}$, where the superscripts refer to the row and column of $G(\theta_{ij})$. For example, the Givens matrix $G(\theta_{34})$ is

$$G(\theta_{34}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_{34} & -\sin\theta_{34} \\ 0 & 0 & \sin\theta_{34} & \cos\theta_{34} \end{bmatrix} \tag{7}$$

To be economical in notation, we will write c_{ij} for $\cos\theta_{ij}$ and s_{ij} for $\sin\theta_{ij}$ when we write the Givens matrices so, for example, the above matrix will be written as:

$$G(\theta_{34}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & -s_{34} \\ 0 & 0 & s_{34} & \mathrm{c}_{34} \end{bmatrix} \tag{8}$$

The Given matrices have the property that they are orthogonal so that $G(\theta_{ij})'G(\theta_{ij}) = G(\theta_{ij})G(\theta_{ij})' = I$. The Givens rotation matrix G that we will use to rotate the initial set of orthogonal shocks $(A_0 e_t)$ to form the new set of orthogonal shocks $(G'A_0 e_t)$ is constructed as the product of the six Givens matrices i.e. as:

$$G = [G(\theta_{12})G(\theta_{13})G(\theta_{14})][G(\theta_{23})G(\theta_{24})][G(\theta_{34})] \tag{9}$$

and it has the property that $G'G = GG' = I$.

The matrix G has the form:

$$G = \begin{bmatrix} c_{12}c_{13}c_{14} & * & * & * \\ s_{12}c_{13}c_{14} & * & * & * \\ s_{13}c_{14} & * & * & * \\ s_{14} & * & * & * \end{bmatrix} \tag{10}$$

The first column of G depends only on the angles θ_{12}, θ_{13} and θ_{14}. Moreover, it can be shown that the first column of G is the same as the first column of $[G(\theta_{12})G(\theta_{13})G(\theta_{14})]$. Now define

$$\bar{G} = [G(\theta_{23})G(\theta_{24})][G(\theta_{34})] \tag{11}$$

It can be established that:

$$\bar{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{23}c_{24} & * & * \\ 0 & s_{23}c_{24} & * & * \\ 0 & s_{24} & * & * \end{bmatrix} \tag{12}$$

The first column of $\bar{G}$ is the unit basis vector and the second column depends only on the angles θ_{23} and θ_{24}. It can be shown that the first two columns of $\bar{G}$ are the same as the first two columns of $[G(\theta_{23})G(\theta_{24})]$. Finally, Eq. (8) shows $G(\theta_{34})$ from which it can be seen that the first two columns are unit basis vectors and the third column depends only on the angle θ_{34}.

We will show how identification of the structural shocks in Peersman's model can be implemented using Givens rotation matrices under either a full set of parametric restrictions, a full set of sign restrictions or by a combination of parametric and sign restrictions. We now turn to the first case of full parametric identification of the shocks.

4. Identification by parametric restrictions

Peersman identified the structural shocks first by parametric restrictions alone and then by sign restrictions alone and compared the impulse responses from both methods. In this section, we show how to implement the parametric restrictions which Peersman utilised by means of Givens rotations. The parametric restrictions identify the shocks as either a monetary policy (MP) shock, an aggregate demand (AD) shock, an aggregate supply (AS) shock or an oil price (OP) shock.

In a four variable model, six parametric restrictions are required to identify the structural shocks. This is the order condition for identification of the shocks. Peersman utilised two long-run zero restrictions and four contemporaneous zero restrictions and they are:

R1: The MP shock has a zero long-run effect on output

R2: The MP shock has a zero contemporaneous effect on output

R3: The MP shock has a zero contemporaneous effect on oil prices

R4: The AD shock has a zero long-run effect on output

R5: The AD shock has a zero contemporaneous effect on oil prices

R6: The AS shock has a zero contemporaneous effect on oil prices

The rank condition for identification of the structural shocks is satisfied here because the shocks for some order have 0-1-2-3 restrictions placed on them. Here the order is 3-2-1-0 i.e. three for the MP shock, two for the AD shock, one for the AS shock and zero for the OP shock. This is a result from Rubio-Ramírez, Waggoner and Zha (2010) who derive the conditions for rank identification in SVARS for a wide range of situations, and this is their result for the situation here.[1]

As there are three restrictions on the MP shock, we will treat the first orthogonal shock as the MP shock because there are three parameters in the first column of the matrix G that are to be determined. In Eq. (5), which shows the long-run response of the variables to the shocks, denote the

[1] For example, if the shocks for some order had 1-2-2-1 restrictions on them, the SVAR is not identified because the rank condition is violated. The order condition is satisfied but it is only necessary and is not sufficient for identification.

typical element of $D(1)A_0^{-1}$ as b_{ij}. The elements of $D(1)A_0^{-1}$ are obtained from estimation of the reduced-form VAR. The first restriction on the MP shock i.e. R1, is obtained by multiplying the second row of $D(1)A_0^{-1}$ with the first column in G, shown in Eq. (10), and setting the expression to zero to obtain:

$$b_{21}c_{12}c_{13}c_{14} + b_{22}s_{12}c_{13}c_{14} + b_{23}s_{13}c_{14} + b_{24}s_{14} = 0 \tag{13}$$

Divide both sides of Eq. (13) by $c_{12}c_{13}c_{14}$ to obtain:

$$b_{21} + b_{22}t_{12} + b_{23}\frac{t_{13}}{c_{12}} + b_{24}\frac{t_{14}}{c_{12}c_{13}} = 0 \tag{14}$$

where we have used the result that $\tan(\theta_{ij}) = \dfrac{\sin(\theta_{ij})}{\cos(\theta_{ij})}$ i.e. in current notation $t_{ij} = \dfrac{s_{ij}}{c_{ij}}$.

In Eq. (6), which shows the contemporaneous response of the variables to the shocks, denote the typical element of A_0^{-1} by a_{ij}, and these are known from the decomposition of Ω. The second restriction on the MP shock, namely R2, is obtained by multiplying the second row of A_0^{-1} with the first column of G, and setting the expression to zero. Then divide both sides by $c_{12}c_{13}c_{14}$ to obtain:

$$a_{21} + a_{22}t_{12} + a_{23}\frac{t_{13}}{c_{12}} + a_{24}\frac{t_{14}}{c_{12}c_{13}} = 0 \tag{15}$$

The third restriction on the MP shock, i.e. R3, is obtained similarly, i.e. by multiplying the first row of A_0^{-1} with the first column of G, to obtain:

$$a_{11} + a_{12}t_{12} + a_{13}\frac{t_{13}}{c_{12}} + a_{14}\frac{t_{14}}{c_{12}c_{13}} = 0 \tag{16}$$

Eqs. (14), (15) and (16) constitute a linear system of equations which can solved for:

$$t_{12} = f^{1}(a_{ij}, b_{ij}) \tag{17}$$

$$\frac{t_{13}}{c_{12}} = f^{2}(a_{ij}, b_{ij}) \tag{18}$$

$$\frac{t_{14}}{c_{12}c_{13}} = f^{3}(a_{ij}, b_{ij}) \tag{19}$$

where each solution is a function of the set of parameters i.e. the a_{ij} and b_{ij} that appear in the system. From Eq. (17), we obtain the estimated value of θ_{12} as:

$$\hat{\theta}_{12} = arc\tan[f^{1}(a_{ij}, b_{ij})] \tag{20}$$

Once we have found $\hat{\theta}_{12}$, we have also found c_{12} which we write as $c_{12}(\hat{\theta}_{12})$, i.e. $\cos\hat{\theta}_{12}$. Then we can find $\hat{\theta}_{13}$ from Eq. (18) as:

$$\hat{\theta}_{13} = arc\tan[c_{12}(\hat{\theta}_{12}) f^2(a_{ij}, b_{ij})] \tag{21}$$

We then find $\hat{\theta}_{14}$ from Eq. (19) as:

$$\hat{\theta}_{14} = arc\tan[c_{12}(\hat{\theta}_{12}) c_{13}(\hat{\theta}_{13}) f^3(a_{ij}, b_{ij})] \tag{24}$$

Having found these, we calculate the matrix $[G(\hat{\theta}_{12})G(\hat{\theta}_{13})G(\hat{\theta}_{14})]$, which we denote as:

$$G^1 = [G(\hat{\theta}_{12})G(\hat{\theta}_{13})G(\hat{\theta}_{14})] \tag{25}$$

Write Eq. (5) as:

$$C(1) = D(1)A_0^{-1}G^1\overline{G} \tag{26}$$

where $\overline{G}$ is shown by Eq. (11) and denote the typical element of $D(1)A_0^{-1}G^1$ as bb_{ij}, which we have now all found. Similarly, write Eq. (6) as:

$$C(0) = A_0^{-1}G^1\overline{G} \tag{27}$$

and denote the typical element of $A_0^{-1}G^1$ as aa_{ij}, which are all obtained. We note that post-multiplication of the matrix in Eq. (26) and in Eq. (27) by $\overline{G}$ will leave the first column unchanged thereby leaving the restrictions on the first shock unaltered. Now there are two restrictions on the AD shock and as there are two free parameters in the second column of $\overline{G}$, which are θ_{23} and θ_{24} (see Eq. (12)), we nominate the second orthogonal shock as the AD shock. Then the restriction R4, namely that the AD shock has a zero long-run effect on output, is obtaining by multiplying the second row of $D(1)A_0^{-1}G^1$ by the second column of $\overline{G}$, and setting the resulting expression to zero. After dividing both sides by $c_{23}c_{24}$, we obtain:

$$bb_{22} + bb_{23}t_{23} + bb_{24}\frac{t_{24}}{c_{23}} = 0 \tag{28}$$

Similarly, the restriction that the AD shock has a zero contemporaneous effect on the oil price (R5) can be obtained as:

$$aa_{12} + aa_{13}t_{23} + aa_{24}\frac{t_{24}}{c_{23}} = 0 \tag{29}$$

Eqs. (28) and (29) constitute a linear system of equations which can be solved for

$$t_{23} = g^1(aa_{ij}, bb_{ij}) \tag{30}$$

$$\frac{t_{24}}{c_{23}} = g^2(aa_{ij}, bb_{ij}) \tag{31}$$

where each solution depends on the parameters in the system. From Eq. (30), the estimated value of θ_{23} is:

$$\hat{\theta}_{23} = arc\tan[g^1(a_{ij}, b_{ij})] \tag{31}$$

and from Eq. (31), the estimated value of θ_{24} is:

$$\hat{\theta}_{24} = arc\tan[c_{23}(\hat{\theta}_{23})g^2(a_{ij}, b_{ij})] \tag{32}$$

We can now calculate the matrix $[G(\hat{\theta}_{23})G(\hat{\theta}_{24})]$ and we know that the first two columns of this matrix are the same as the first two columns of $\overline{G}$. Now let

$$G^2 = [G(\hat{\theta}_{23})G(\hat{\theta}_{24})] \tag{33}$$

and write Eq. (27) as:

$$C(0) = A_0^{-1}G^1G^2G(\theta_{34}) \tag{34}$$

We treat the third orthogonal shock as the AS shock as there is only one parametric restriction placed on it and there is only one free parameter in the matrix $G(\theta_{34})$. We emphasize that post-multiplication of the matrix $A_0^{-1}G^1G^2$ by $G(\theta_{34})$ will leave the first two columns unchanged so that the restrictions already imposed on the first two orthogonal shocks are unaltered. Denote the typical element of the matrix $A_0^{-1}G^1G^2$ as aaa_{ij}. The restriction on the AS shock, i.e. R6, namely, that it has a zero contemporaneous effect on the oil price, is obtained by multiplying the first row of $A_0^{-1}G^1G^2$ with the third column of $G(\theta_{34})$. Setting the resulting expression to zero and dividing both sides by c_{34} we obtain:

$$aaa_{13} + aaa_{14}t_{34} = 0 \tag{35}$$

Solve this equation to get the estimate of θ_{34} as:

$$\hat{\theta}_{34} = arc\tan(\frac{-aaa_{13}}{aaa_{14}}) \tag{36}$$

Now we can calculate $G(\hat{\theta}_{34})$ and find all the elements of G, viz, $G = G^1G^2G(\hat{\theta}_{34})$.

This method produces estimates for the all of the free parameters (angles) when it is the case that the SVAR is identified by parametric restrictions alone. Under full parametric identification, there are six restrictions in Peersman's model, and they are implemented by solving for the six θ_{ij} parameters in the Givens rotation matrices. We now turn to identification of the shocks by sign restrictions alone, and in this case, the six θ_{ij} parameters in the Givens rotation matrices are generated randomly.

5. Identification by sign restrictions

Under full sign restrictions, all of the parameters in the Givens rotation matrix are generated. For each generated Givens matrix, a new set of orthogonal shocks are formed to which impulse responses are calculated and are judged for either acceptance or rejection by the sign restrictions.

This procedure is repeated until a predetermined number of sets of impulse responses are accepted and that may take many thousands of repetitions. Each repetition involves another draw of the θ_{ij} parameters in the Givens rotation matrix, and in Peersman's model as we have seen, there are six of them. In this section, we present a method to draw the six θ_{ij} parameters to form the Givens rotation matrix. We show that when the parameters are drawn in this way, the Givens rotation matrix is equivalent to the orthogonal matrix produced by the QR decomposition method.

First, consider the matrix $[G(\theta_{12})G(\theta_{13})G(\theta_{14})]$ which we know has the same first column as G, which is shown in Eq. (10). Now generate a (4x1) vector Z with each element z_i randomly drawn from a $N(0,1)$ distribution and normalize each element by the norm of Z. The normalized vector is W, where $w_i = z_i / \sqrt{z_1^2 + z_2^2 + z_3^2 + z_4^2}$ for $i = 1,2,3,4.$ Then equate each element in the first column of $[G(\theta_{12})G(\theta_{13})G(\theta_{14})]$ with the corresponding element in the vector W. By Eq. (10) this produces:

$$c_{12}c_{13}c_{14} = w_1 \tag{37}$$

$$s_{12}c_{13}c_{14} = w_2 \tag{38}$$

$$s_{13}c_{14} = w_3 \tag{39}$$

$$s_{14} = w_4 \tag{40}$$

Divide Eq. (38) by Eq. (37) to get:

$$t_{12} = \frac{w_2}{w_1} \tag{41}$$

which gives

$$\hat{\theta}_{12} = arc\tan\left(\frac{w_2}{w_1}\right) \tag{42}$$

Similarly, divide Eq. (39) by Eq. (38)

$$t_{13} = s_{12}\frac{w_3}{w_2} \tag{43}$$

from which we get

$$\hat{\theta}_{13} = arc\tan\left(s_{12}(\hat{\theta}_{12})\frac{w_3}{w_2}\right) \tag{44}$$

Finally, divide Eq. (40) by Eq. (39) to get

$$t_{14} = s_{13}\frac{w_4}{w_3} \tag{45}$$

from which we have:

$$\hat{\theta}_{14} = arc\tan\left(s_{13}(\hat{\theta}_{13})\frac{w_4}{w_3}\right) \tag{46}$$

Second, consider the matrix $[G(\theta_{23})G(\theta_{24})]$, which we know has the same second column as $\bar{G}$, which is shown in Eq. (12). Now generate a (3x1) vector $\tilde{Z}$ with each element $\tilde{z}_i$ randomly drawn from a $N(0,1)$ distribution and normalize each element by the norm of $\tilde{Z}$. The normalized vector is $\tilde{W}$, where $\tilde{w}_i = \tilde{z}_i / \sqrt{\tilde{z}_1^2 + \tilde{z}_2^2 + \tilde{z}_3^2}$ for $i = 1,2,3$. Then equate each (non-zero) element in the second column of $[G(\theta_{23})G(\theta_{24})]$ with the corresponding element in the vector $\tilde{W}$. By Eq. (12) this produces:

$$c_{23}c_{24} = \tilde{w}_1 \tag{47}$$

$$s_{23}c_{24} = \tilde{w}_2 \tag{48}$$

$$s_{24} = \tilde{w}_3 \tag{49}$$

Solving as before, we obtain:

$$\hat{\theta}_{23} = arc\tan\left(\frac{\tilde{w}_2}{\tilde{w}_1}\right) \tag{50}$$

$$\hat{\theta}_{24} = arc\tan\left(s_{23}(\hat{\theta}_{23})\frac{\tilde{w}_3}{\tilde{w}_2}\right) \tag{51}$$

Third, consider the matrix $G(\theta_{34})$. Generate a (2x1) vector $\bar{Z}$ with each element $\bar{z}_i$ randomly drawn from a $N(0,1)$ distribution and normalize each element by the norm of $\bar{Z}$. The normalized vector is $\bar{W}$, where $\bar{w}_i = \bar{z}_i / \sqrt{\bar{z}_1^2 + \bar{z}_2^2}$ for $i = 1,2$. Then equate each (non-zero) element in the third column of $G(\theta_{34})$ with the corresponding element in the vector $\bar{W}$. By Eq. (8) this produces $c_{34} = \bar{w}_1$ and $s_{34} = \bar{w}_2$ from which we obtain:

$$\hat{\theta}_{34} = arc\tan\left(\frac{\bar{w}_2}{\bar{w}_1}\right) \tag{52}$$

Having obtained estimates for all of the θ_{ij} parameters, the Givens matrix G is calculated (see Eq. (9)), and the new set of orthogonal shocks is calculated and the impulse responses are obtained. The procedure is repeated by taking another set of draws from the standard normal distribution to calculate another Givens matrix, which is used to form another set of orthogonal shocks for which impulse responses are obtained.

This method of generating the Givens matrix will produce an orthogonal matrix which is the same as the orthogonal matrix produced by the QR decomposition. The QR decomposition involves forming the matrix W, which is (4x4) in our application, by drawing each of its columns randomly from a $N(0, I_4)$ distribution, and performing the QR factorization of W on each draw. This factorization is $W = QR$, where Q is an orthogonal matrix and R is upper triangular. The Givens matrix G, formed in the manner described above, will be the practically same as the matrix Q formed from the QR factorization of $W.$ To see this, Figure 1 shows the empirical distribution of each element in the G and Q matrices, respectively, generated from one hundred thousand of them. The figure shows that the empirical distributions of each element overlap and are practically indistinguishable. There are other ways to find the θ_{ij} parameters in the Givens rotation. Peersman (2005) draws each of the six θ_{ij} parameters directly from a uniform distribution over the interval $[0,\pi]$ to form $G.$ In this case, the empirical distributions of some of the respective elements in the G and Q matrices do not coincide so that this method of generating the Givens rotation produces an orthogonal matrix which is not the same as that produced by the QR factorization.

The sign restrictions on the impulse responses utilised by Peersman (2005) are shown in Table 1 below.

Table 1. Sign Restrictions

Shock\Variable	Oil Price	Output	Consumer Prices	Interest rate
OP	≥0	≤0	≥0	≥0
AS	?	≥0	≤0	≤0
AD	≥0	≥0	≥0	≥0
MP	≤0	≤0	≤0	≥0

In the table, the designation "≥0" denotes a non-negative response so that the variable does not fall in response to the shock while "≤0" denotes a non-positive response so that the variable does not rise in response to the shock. The designation "?" denotes an unrestricted response. Peersman does not sign restrict the response of oil prices to an AS shock. In order to separate the OP shock from the AS shock, the shock which has the largest contemporaneous effect on the oil price is treated as the OP shock. This is a size restriction. The sign restrictions are standard. For example, they rule out 'output' and 'price' puzzles so that in response to an MP shock which raises the interest rate, oil prices, output, and consumer prices cannot rise. The number of quarters over which the sign restrictions are applied to the impulse responses is four quarters for output and consumer prices and one quarter for the oil price and the interest rate.

We now turn to identification of the orthogonal shocks under a combination of parametric and sign restrictions.

6. Identification by combining parametric and sign restrictions.

In this section, we consider five different identifications of the orthogonal shocks which involve some combination of parametric and sign restrictions.

6.1 Identification A

We restrict the first two orthogonal shocks to have a zero long-run effect on output. This removes the sign restrictions on the response of output to the first two orthogonal shocks shown in Table 1. The sign restrictions that are left are sufficient to separate the first two orthogonal shocks as either the MP or AD shock. The other two orthogonal shocks are separate from these as they can have a long-run effect on output and they are separated as either the AS or the OP shock by the size restriction.

The restriction that the first orthogonal shock has a zero long-run effect on output is given by Eq. (14) which is reproduced below:

$$b_{21} + b_{22}t_{12} + b_{23}\frac{t_{13}}{c_{12}} + b_{24}\frac{t_{14}}{c_{12}c_{13}} = 0$$

As there is only one parametric restriction on the first orthogonal shock, we generate the values for θ_{12} and θ_{13}, substitute these into Eq. (14) and solve for θ_{14}. To generate values for the two parameters, we follow an analogous procedure to that in the preceding section. Generate a (3x1) vector Z with each element z_i randomly drawn from a $N(0,1)$ distribution and normalize each element by the norm of Z. The normalized vector is W, where $w_i = z_i / \sqrt{z_1^2 + z_2^2 + z_3^2}$ for $i = 1,2,3$. Then equate the first three elements in the first column of $[G(\theta_{12})G(\theta_{13})G(\theta_{14})]$ with the corresponding element in the vector W. This will produce the system of equations given by Eq. (37), Eq. (38) and Eq. (39) which can be solved for $\hat{\theta}_{12}$ and $\hat{\theta}_{13}$ given in Eq. (42) and Eq. (44), respectively. Substitute these values into Eq. (14) to get:

$$b_{21} + b_{22}t_{12}(\hat{\theta}_{12}) + b_{23}\frac{t_{13}(\hat{\theta}_{13})}{c_{12}(\hat{\theta}_{12})} + b_{24}\frac{t_{14}}{c_{12}(\hat{\theta}_{12})c_{13}(\hat{\theta}_{13})} = 0 \tag{53}$$

Then solve for t_{14} from which we obtain $\hat{\theta}_{14}$ as:

$$\hat{\theta}_{14} = arc\tan\left[\frac{-c_{12}(\hat{\theta}_{12})c_{13}(\hat{\theta}_{13})}{b_{24}}\left(b_{21} + b_{22}t_{12}(\hat{\theta}_{12}) + b_{23}\frac{t_{13}(\hat{\theta}_{13})}{c_{12}(\hat{\theta}_{12})}\right)\right] \tag{54}$$

The restriction that the second orthogonal shock has a zero long-run impact on output is given by Eq. (28) which is reproduced below:

$$bb_{22} + bb_{23}t_{23} + bb_{24}\frac{t_{24}}{c_{23}} = 0$$

We generate the value for θ_{23} and solve from this equation for the value of θ_{24}. We start by generating a (2x1) vector $\tilde{Z}$ with each element $\tilde{z}_i$ randomly drawn from a $N(0,1)$ distribution and normalize each element by the norm of $\tilde{Z}$. The normalized vector is $\tilde{W}$, where $\tilde{w}_i = \tilde{z}_i / \sqrt{\tilde{z}_1^2 + \tilde{z}_2^2}$ for $i = 1,2$. Then equate the second and third elements in the second column of $[G(\theta_{23})G(\theta_{24})]$ with the corresponding element in the vector $\tilde{W}$. This produces the two equations given by Eq. (47) and Eq. (48), respectively, which are solved for $\hat{\theta}_{23}$, shown by Eq. (50). Substitute $\hat{\theta}_{23}$ into Eq. (28) and solve for $\hat{\theta}_{24}$ to get:

$$\hat{\theta}_{24} = arc\tan\left[\frac{-c_{23}(\hat{\theta}_{23})}{bb_{24}}\left(bb_{22} + bb_{23}t_{23}(\hat{\theta}_{23})\right)\right] \tag{55}$$

Finally, we generate the value for θ_{34} using the method described earlier that produces the value of it given by Eq. (52).

Here we solve for the parameter θ_{14} after generating the values for θ_{12} and θ_{13}, and solve for the parameter θ_{24} after generating the value for θ_{23}. The parameter θ_{34} is generated. The number of parameters which are solved for is equal to the number of parametric restrictions while the number of remaining parameters is equal to the number of parameters that are generated. Once the values of all the parameters are obtained, the Givens rotation matrix is obtained from Eq. (9).

6.2 Identification B

We extend Identification A by adding a further parametric restriction which is that the third orthogonal shock does not have a contemporaneous effect on the interest rate. This restriction is similar to that used by Cushman and Zha (1997) and Kim and Roubini (2000). The third parametric restriction separates the third and fourth orthogonal shocks so that there is no need for the size restriction, and when the responses to the third shock satisfy the remaining sign restrictions, it is interpreted as the AS shock.

Instead of generating the value of the θ_{34} parameter as we did for Identification A, we obtain its value from the equation for the restriction on the third orthogonal shock. Analogous to the development following Eq. (34), this restriction gives the estimate of θ_{34} as:

$$\hat{\theta}_{34} = arc\tan(\frac{-aaa_{43}}{aaa_{44}}) \tag{56}$$

We now calculate G and proceed to obtain the impulse responses.

6.3 Identification C

As before we restrict the first two orthogonal shocks to have a zero long-run effect on output and we further restrict the first orthogonal shock to have a zero contemporaneous effect on output, so there are now three parametric restrictions. These parametric restrictions are used in place of the sign restrictions on the response of output to the first two orthogonal shocks, and the sign restrictions that remain are sufficient to separate them as either the MP shock or the AD shock. The other two orthogonal shocks are separate from these by the parametric restrictions, and are separated from each other as either the AS or the OP shock by the size restriction.

The restrictions that first orthogonal shock has a zero long-run and contemporaneous effect on output are shown by Eq. (14) and Eq. (15), respectively. As there are only two parametric restrictions on the first orthogonal shock, we generate the value for θ_{12} and solve for θ_{13} and θ_{14} from Eq. (14) and Eq. (15). To generate the value for θ_{12}, we generate a (2x1) vector Z with each element z_i randomly drawn from a $N(0,1)$ distribution and normalize each element by the norm of Z. The normalized vector is W, where $w_i = z_i / \sqrt{z_1^2 + z_2^2}$ for $i = 1,2$. Then equate the first two elements in the first column of $[G(\theta_{12})G(\theta_{13})G(\theta_{14})]$ with the corresponding element in the vector W. This will produce the two equations given by Eq. (37) and Eq. (38) which can be solved for $\hat{\theta}_{12}$ given in Eq. (42). Substitute into Eq. (14) and Eq. (15) to get, respectively:

$$b_{21}+b_{22}t_{12}(\hat{\theta}_{12})+b_{23}\frac{t_{13}}{c_{12}(\hat{\theta}_{12})}+b_{24}\frac{t_{14}}{c_{12}(\hat{\theta}_{12})c_{13}}=0 \tag{57}$$

$$a_{21}+a_{22}t_{12}(\hat{\theta}_{12})+a_{23}\frac{t_{13}}{c_{12}(\hat{\theta}_{12})}+a_{24}\frac{t_{14}}{c_{12}(\hat{\theta}_{12})c_{13}}=0 \tag{58}$$

Eq. (57) and Eq. (58) are a linear system of two equations in the two unknown t_{13} and t_{14}/c_{13}. The solution can be characterised as:

$$t_{13}=h^{1}(t_{12}(\hat{\theta}_{12}),c_{12}(\hat{\theta}_{12}),a_{ij},b_{ij}) \tag{59}$$

$$\frac{t_{14}}{c_{13}}=h^{2}(t_{12}(\hat{\theta}_{12}),c_{12}(\hat{\theta}_{12}),a_{ij},b_{ij}) \tag{60}$$

where the a_{ij} and b_{ij} are those in Eq. (57) and Eq. (58). We solve these two equations recursively as before to get:

$$\hat{\theta}_{13}=arc\tan[h^{1}(t_{12}(\hat{\theta}_{12}),c_{12}(\hat{\theta}_{12}),a_{ij},b_{ij})] \tag{61}$$

$$\hat{\theta}_{14}=arc\tan[c_{13}(\hat{\theta}_{13})h^{2}(t_{12}(\hat{\theta}_{12}),c_{12}(\hat{\theta}_{12}),a_{ij},b_{ij})] \tag{62}$$

We now proceed to impose the parametric restriction that the second orthogonal shock has a zero long-run effect on output. This development follows exactly what was done for identification A. Under Identification C, we solve for three of the parameters, namely, θ_{13}, θ_{14} and θ_{24} as there are three parametric restrictions and three of the parameters, namely, θ_{12}, θ_{23} and θ_{34} are obtained by the generation method.

6.4 Identification D

Fisher, Huh and Pagan (2016) re-considered Peersman's model by replacing the oil price in the vector of variables by the relative oil price, defined as the difference between the log oil price and the log of consumer prices. They imposed four long-run parametric restrictions and two contemporaneous restrictions. The long-run restrictions were that the MP and AD shocks had a zero long-run effect on output and relative prices.

For Identification D, we impose an equivalent set of long-run restrictions. They are that the first and the second orthogonal shocks have a zero long-run effect on output and that each changes the oil price and consumer prices by the same amount in the long-run. The parametric restrictions only replace the sign restrictions on the response of output to the first two orthogonal shocks. The sign restrictions on the responses of the other variables to the first two orthogonal shocks have to be retained as otherwise these two shocks cannot be separated as either the MP or AD shock. The other two orthogonal shocks are separate from the first two as no parametric restrictions are placed on them, and they are separated from each other by the size restriction.

The restriction that the first orthogonal shock has a zero long-run effect on output is given by Eq. (14) which is reproduced below:

$$b_{21}+b_{22}t_{12}+b_{23}\frac{t_{13}}{c_{12}}+b_{24}\frac{t_{14}}{c_{12}c_{13}}=0$$

The restrictions that the first shock changes both oil prices and consumer prices by k in the long-run are, respectively:

$$b_{11} + b_{12}t_{12} + b_{13}\frac{t_{13}}{c_{12}} + b_{14}\frac{t_{14}}{c_{12}c_{13}} = k \tag{63}$$

$$b_{31} + b_{32}t_{12} + b_{33}\frac{t_{13}}{c_{12}} + b_{34}\frac{t_{14}}{c_{12}c_{13}} = k \tag{64}$$

By equating Eq. (63) with Eq. (64), we obtain the restriction that the first shock changes the oil price and the consumer price by the same amount in the long-run and it is:

$$(b_{11} - b_{31}) + (b_{12} - b_{32})t_{12} + (b_{13} - b_{33})\frac{t_{13}}{c_{12}} + (b_{14} - b_{34})\frac{t_{14}}{c_{12}c_{13}} = 0 \tag{65}$$

We generate a value for θ_{12} using the method as before, and we denote it as $\hat{\theta}_{12}$. We then solve Eq. (14) and Eq. (65) as before for $\hat{\theta}_{13}$ and $\hat{\theta}_{14}$.

The restriction that the second orthogonal shock has a zero long-run effect on output is given by Eq. (28) which is re-produced below:

$$bb_{22} + bb_{23}t_{23} + bb_{24}\frac{t_{24}}{c_{23}} = 0$$

Following an analogous development as above, the restriction that the second orthogonal shock changes the oil price and the consumer price by the same amount is:

$$(bb_{12} - bb_{32}) + (bb_{13} - bb_{33})t_{23} + (bb_{14} - bb_{34})\frac{t_{24}}{c_{23}} = 0 \tag{66}$$

Eq. (28) and Eq. (66) are two equations in the two unknowns t_{23} and t_{24}/c_{23}. Solve for both as before and obtain $\hat{\theta}_{23}$ and $\hat{\theta}_{24}$. Then generate $\hat{\theta}_{34}$. Having obtain estimates for all the parameters, we can obtain G.

From Eq. (63) or Eq. (64), it can readily be seen that this method can handle the case of non-zero restrictions, which can arise in empirical macro-econometrics.

6.5 Identification E

For Identification E, a third restriction is imposed on the first orthogonal shock in Identification D. The additional restriction is that this shock has a zero contemporaneous effect on output, and this restriction is expressed by Eq. (15). We then solve Eq. (14), Eq. (15) and Eq. (65) for $\hat{\theta}_{12}$, $\hat{\theta}_{13}$ and $\hat{\theta}_{14}$ using the solution method described previously and proceed as we did before in Identification D with respect to the other shocks. The three restrictions on the first orthogonal shock and the two on the second orthogonal shock identify the former as the MP shock and the latter as the AD shock. No sign restrictions are required to separate the MP and AD shocks here. As before the third and fourth orthogonal shocks are separated as either the OP or AS shock by the size restriction.

7. Results

We now turn to the results from Peersman's model for each of the identification schemes we considered earlier. We estimated the VAR using Peersman's original data which is for the United States (US) and covers the period 1980:Q1 to 2002:Q2. The data was obtained from the *Journal of Applied Econometrics* data archive. Following Peersman, the VAR was specified with three lags and a constant and a time trend were included in each equation. Peersman allowed for estimation uncertainty in the VAR. Following his approach, we estimate the parameters of the VAR in a Bayesian framework. The prior for the VAR coefficients and Ω^{-1} is Normal-Wishart. For these priors, the posterior for the VAR coefficients and Ω^{-1} is also Normal-Wishart. We report the impulse responses for the levels of the series.

7.1 Full parametric identification

Figure 2 shows the impulse responses from the full parametric identification of the shocks together with 84th and 16th percentile error bands. The impulse responses replicate those in Figure 1(a) of Peersman. The six parametric restrictions clearly show up in the impulse responses. We make the following observations. In response to a positive AD shock, oil prices and consumer prices rise permanently but the permanent increase in oil prices is much larger than the permanent increase in consumer prices. Output rises immediately and returns to its level prior to the shock by eight quarters. For the MP shock, which raises the interest rate, oil prices and consumer prices fall permanently and the permanent decrease in oil prices is much larger than the permanent decrease in consumer prices. Output rises marginally on impact so there is a slight output puzzle but there is no price puzzle. Output then falls and after three quarters gradually returns to it level prior to the shock. The AD and MP shocks have very little impact on output in the long-run.

7.2 Full signs identification

We now turn to the results from full sign identification of the shocks. For each generated Givens matrix G, we calculate the impulse responses to each of the orthogonal shocks and if the full set of responses satisfy the sign restrictions for the shocks to be MP, AD, AS and OP shocks in any order, they are retained. If not, the full set of responses are discarded. We continue to generate the Givens matrix and calculate the impulse responses until 1000 sets of responses are retained. We arrange the accepted responses into ascending order at each horizon and find the median of the responses. The medians are connected point-wise across horizons to form the median impulse response. We also find the 84th and 16th percentiles of the accepted responses at each horizon and connect them point-wise across horizons to form the percentile responses.

It is important to note that the median response at each horizon is likely to come from a different Givens matrix G i.e. from a different set of the six θ_{ij} parameters. Because the median response at each horizon likely comes from a different rotation, the median impulse response function does not correspond to a single rotation. Fry and Pagan (2011) refer to this as the multiple models problem. They propose a metric to find the set of impulse responses, from among the set of all accepted impulse responses, which comes 'closest' to the median responses. This set corresponds to a single rotation i.e. to a single G matrix, and the responses are known as the median-target responses, which we also report.

Figure 3 reports the median (and median-target) responses, together with the percentile bands, for the identification from full sign restrictions. The median impulse responses replicate those reported in Figure 2(a) of Peersman. As observed by Peersman, there are some interesting differences between the results from this case and the full parametric case. Under signs, there is a large contemporaneous impact of the AD and MP shocks on the price of oil which suggests that the contemporaneous zero restriction for oil prices may be unduly restrictive. The median response of the oil price to the AS shock is also large but it is quite uncertain as the percentile responses cover the zero axis. The other noticeable difference is that impact response of output to the MP shock is substantial. Output falls markedly on impact, rebounds somewhat the next quarter and then falls further for several more quarters before returning gradually to its level prior to the shock. There is no evidence of an output puzzle here. The median responses of output to the AD and MP shocks at long horizons are both close to zero, suggesting monetary neutrality in the long-run.

7.3 Combining parametric and sign restrictions

Figure 4 shows the results under identification A for which there are two parametric restrictions, namely, that the MP and AD shocks have a zero long-run effect on output. We remark on the following results. The median response shows that output falls slightly on impact in response to the MP shock. This response is quite uncertain as the zero impact response is within the two percentile responses. This is in contrast to the full sign case, where output falls markedly on impact and the region between the two percentile impact responses is narrow and does not include the zero impact response. The median response to the AD shock shows that consumer prices rise permanently in the long-run, whereas under the full sign case, the median response is close to zero at long horizons.

Figure 5 shows the results under identification B. The parametric restrictions are those for identification A along with the additional restriction that the AS shock does not have a contemporaneous effect on the interest rate. This restriction is motivated by the observation that the central bank may not observe the effect on output of a shock in the current quarter because it may take several quarters for the latest GDP data to be released. As a consequence, it does not adjust the interest rate within the quarter to an AS shock. This parametric restriction is clearly seen in the figure as the impact response of the interest rate to the AS shock is zero with no uncertainty. The median impact response of output to the MP shock is indistinguishable from zero, whereas in the full sign case, output falls on impact considerably. Figure 6 shows the results for identification C. The two long-run parametric restrictions are maintained but now the third parametric restriction is that the MP shock has a zero contemporaneous effect on output. Under this identification, the slight output puzzle that we saw under full parametric identification re-emerges. Interestingly, the median response shows a considerable fall in the interest rate on impact following an AS shock, which is also seen under identification A and full signs identification.

Figure 7 shows the results under identification D where the MP and AD shocks have a zero long-run effect on output and where they change the oil and consumer prices by the same amount in the long-run. These restrictions impose long-run monetary neutrality and they clearly show up in the median responses. In response to the MP and AD shocks, oil prices change by much less in the long-run under this identification than they did under previous identifications. In response to the AD shock, the oil price rises by less than 0.1 percentage points in the long-run whereas under the previous identifications the oil price rose by between 3 and 8 percentage points. In response to the MP shock oil prices fall by about 0.4 percentage points in the long-run whereas in the earlier identifications they fell by between 2 and 6 percentage points. The median long-run response of consumer prices is small, less than 0.1 percentage points, which is similar to that obtained under full sign identification, but is considerably smaller than that obtained under the other identifications. The median response of output to the MP shock shows that output falls markedly on impact, which is also seen under full signs identification. Figure 8 shows the results under the four long-run restrictions together with the restriction that the MP shock has a zero contemporaneous effect on output i.e. identification E. The only significant change to the results from identification D is that the slight output puzzle re-emerges.

8. Relationship to the Method of Aries, Rubio-Ramírez and Waggoner (2018)

Aries, Rubio-Ramírez and Waggoner (2018) develop a method to combine sign and parametric restrictions based on the QR decomposition. They show that the zero restrictions on the impulse responses can be converted into linear restrictions on the columns of the orthogonal matrix Q. We briefly describe their method, hereafter referred to as ARW, in the context of identification C which imposes two long-run zero restrictions and one contemporaneous zero restriction. Following the development in Section 2, the impulse responses to the new set of orthogonal shocks is given by:

$$C(L) = D(L)A_0^{-1}Q \tag{67}$$

where Q is an orthogonal matrix. Write $Q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}$ where each q_i is a (4x1) vector orthogonal to the other vectors. As before, let the typical element of $D(1)A_0^{-1}$ be b_{ij} and let the typical element of $D(0)A_0^{-1}$ be a_{ij}, where $D(0) = I$. We treat the shock with the most parametric restrictions placed on it as the first shock.

The ARW method finds the vector q_1 which satisfies the two restrictions on the first shock, namely, that it has neither a contemporaneous nor long-run effect on output. i.e. it finds q_1 such that:

$$R_1 q_1 = 0 \tag{68}$$

where

$$R_1 = \begin{pmatrix} b_{21} & b_{22} & b_{23} & b_{24} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \tag{69}$$

Their method to find q_1 proceeds as follows: (i) Find an orthonormal matrix N_1 whose columns form a basis for the null space of R_1. In this case, N_1 has dimension two so that it is a (4x2) matrix; (ii) draw a (4x1) vector x_1 from the standard normal distribution of $N(0, I_4)$; (iii) apply the Gram-Schmidt method to generate the (4x1) vector q_1, as $q_1 = N_1(N_1'x_1 / \| N_1'x_1 \|)$, where ||•|| is the Euclidian norm.

There is one restriction on the second shock, namely, that it has a zero long-run effect on output. We also require that q_2 be orthogonal to the already constructed q_1 vector. These restrictions can be expressed as:

$$R_2 q_2 = 0 \tag{70}$$

where

$$R_2 = \begin{pmatrix} b_{21} & b_{22} & b_{23} & b_{24} \\ & q_1' & & \end{pmatrix} \tag{71}$$

The method to find q_2 proceeds similarly: (i) Find an orthonormal matrix N_2 whose columns form a basis for the null space of R_2, and this matrix will be (4x2); (ii) draw a (4x1) vector x_2 from the standard normal distribution of $N(0, I_4)$; (iii) apply the Gram-Schmidt method to generate the (4x1) vector q_2, as $q_2 = N_2(N_2' x_2 / \| N_2' x_2 \|)$.

As there are no further parametric restrictions, the ARW method finds q_3 and q_4 so that they are orthogonal to each other and to q_1 and q_2. To find q_3, (i) form $R_3 = \begin{pmatrix} q_1' & q_2' \end{pmatrix}'$ and find an orthonormal matrix N_3 whose columns form a basis for the null space of R_3, in which case it will be (4x2); (ii) draw a (4x1) vector x_3 from the standard normal distribution of $N(0, I_4)$; (iii) apply the Gram-Schmidt method to generate the (4x1) vector q_3, as $q_3 = N_3(N_3' x_3 / \| N_3' x_3 \|)$. Finally, to find q_4, (i) form $R_4 = \begin{pmatrix} q_1' & q_2' & q_3' \end{pmatrix}'$ and find an orthonormal matrix N_4 whose columns form a basis for the null space of R_4, where now it will be (4x1); (ii) draw a (4x1) vector x_4 from the standard normal distribution of $N(0, I_4)$; (iii) apply the Gram-Schmidt method to generate the (4x1) vector q_4, as $q_4 = N_4(N_4' x_4 / \| N_4' x_4 \|)$. The orthogonal Q matrix with the three parameter restrictions imposed is $Q = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}$. The procedure produces another Q matrix for another draw of the four vectors x_1, x_2, x_3 and x_4 and it is repeated many times. Figure 9 shows the results for this identification under the ARW method. The results are very similar to those shown in Figure 6 which were obtained from the Givens rotation method.

The ARW method, in common with the method that utilises the Givens rotations, can impose at most $4 - j$ zero restrictions for each shock $j = 1, 2, 3, 4.$ The ARW method, however, cannot implement non-zero restrictions because it involves calculating matrices whose columns form a basis for the null space of the matrix of parametric restrictions i.e. the parametric restrictions are zero restrictions. For example, the restriction that the first shock changes consumer prices by $k = 3\%$ in the long-run cannot be implemented in the ARW method. This restriction can be implemented under Givens rotations because that method involves solving systems of equations directly. However, that may not be a significant drawback of the ARW method because quantitative economic theory seldom delivers precise long-run predictions such that a variable will increase by x-percent in the long-run. The ARW approach, however, can implement Identification D because the non-zero restrictions on the shocks can be combined to form zero restrictions, analogous to that shown by Eq. (65) and Eq. (66).

9. Conclusion

In this paper, we develop a procedure for implementing parametric and sign restrictions using the properties of Givens rotation matrices. We show how the method works in the context of several different identifications of the shocks in Peersman's model (2005) that utilise both sign and parametric restrictions. We relate this method to the method of Aries, Rubio-Ramírez and Waggoner (2018). While their method is straightforward to implement and may be more computationally efficient, it cannot accommodate non-zero parametric restrictions. In future research, we intend to apply the Givens approach to empirical applications where non-zero parametric restrictions, which may arise from extraneous information, are used in combination with sign restrictions.

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Figure 1

Empirical distribution of the elements in the Givens and Q matrices, where Q is from a QR decomposition

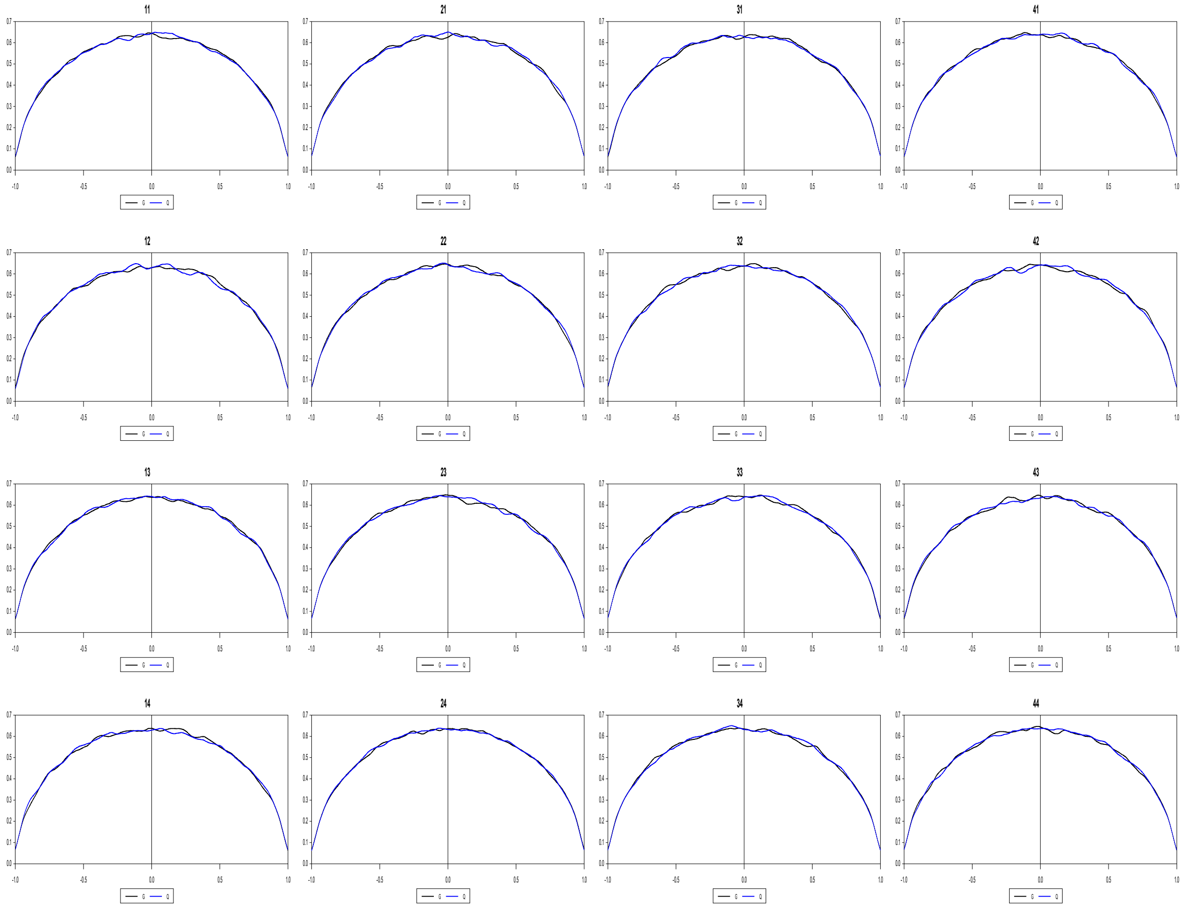

Figure 2

Full parametric identification of Peersman's model: Two long-run and four contemporaneous zero restrictions

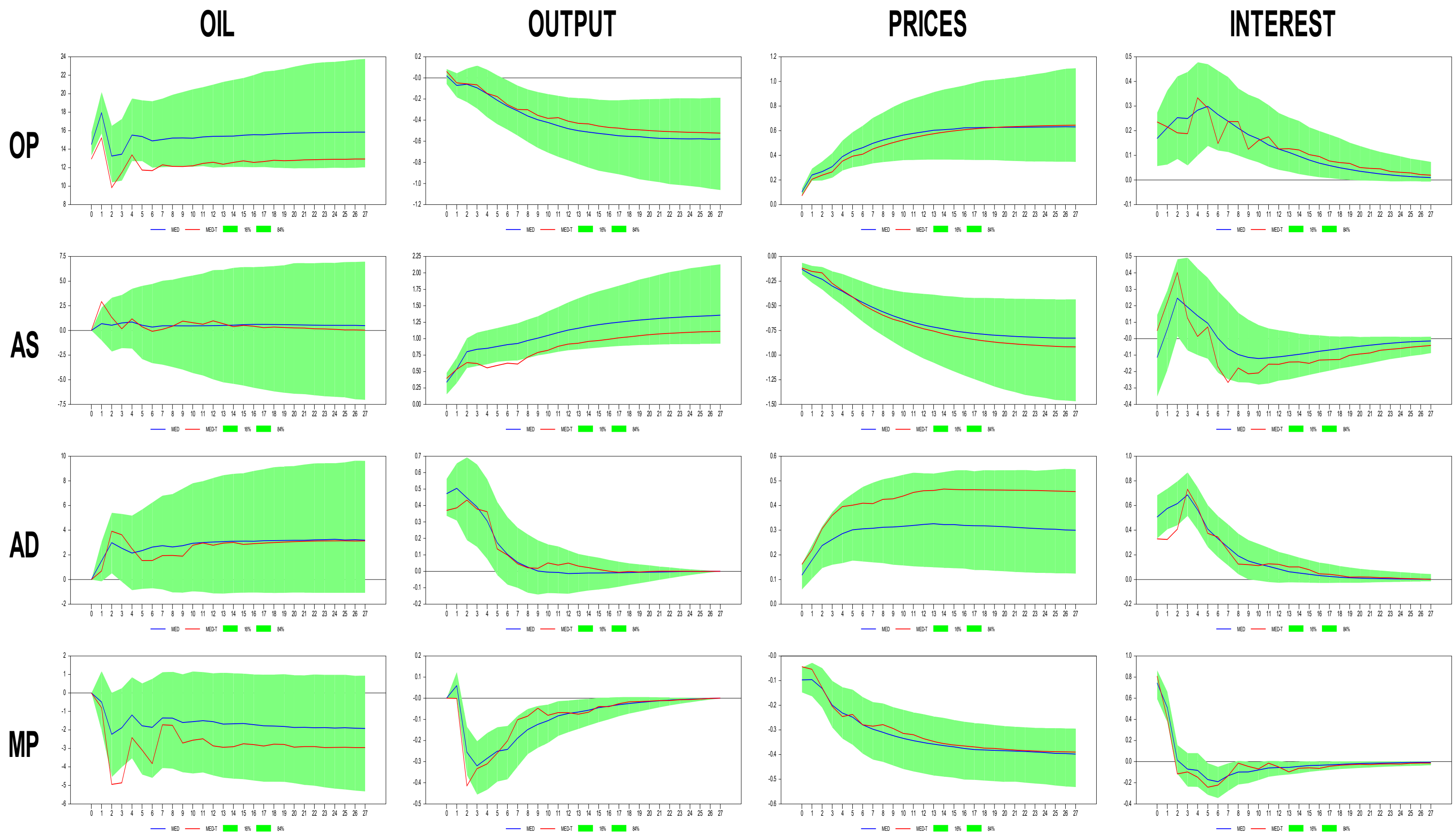

Figure 3

Full sign indentification of Peersman's model. Acceptance rate =0.0316%

Figure 4

Sign and parametric identification of Peersman's model: Identification A - two long-run zero restrictions, acceptance rate = 0.04608%

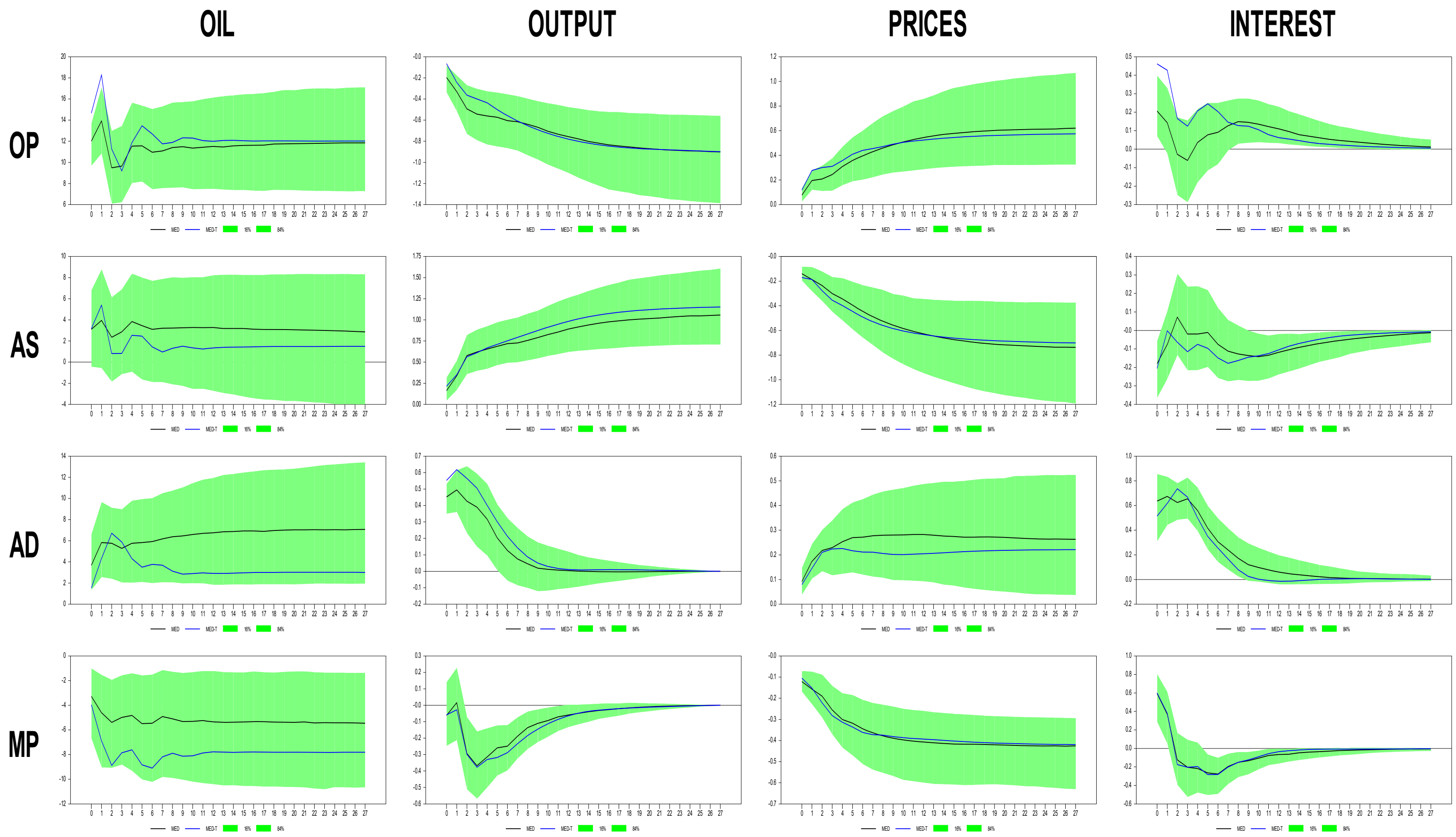

Figure 5

Sign and parametric identification of Peersman's model: Identification B - two long-run zero restrictions and the AS shock has a zero contemporaneous effect on the interest rate, acceptance rate = 0.08837%

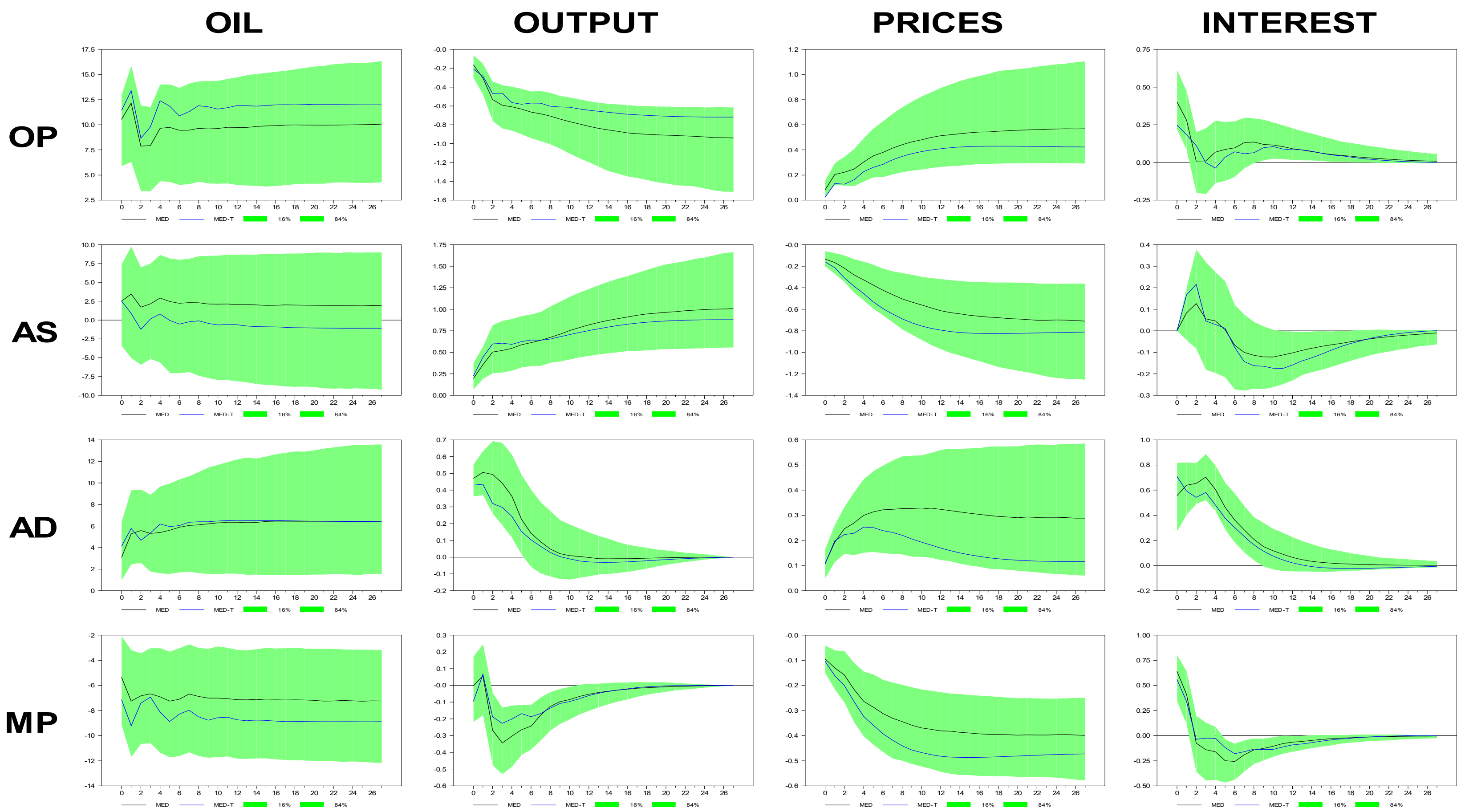

Figure 6

Sign and parametric identification of Peersman's model: Identification C - two long-run zero restrictions and the MP shock has a zero contemporaneous effect on output, acceptance rate = 0.04306%

Figure 7

Sign and parametric identification of Peersman's model: Identification D - Four long-run zero restrictions, acceptance rate = 0.0903%

Figure 8

Sign and parametric identification of Peersman's model: Identification E - Four long-run zero restrictions and the MP shock has a zero contemporaneous effect on output, acceptance rate = 0.4688%

Figure 9

Sign and parametric identification of Peersman's model using Aries et al method: Identification C - two long-run zero restrictions and the MP shock has a zero contemporaneous effect on output, acceptance rate = 0.0659%